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THE LINEARISED EQUATIONS OF MOTION

The equations of motion and equilibrium of an elastic material subjected to small strains were established in the first half of the nineteenth century by Poisson, Navier, Cauchy, Stokes, Green and others. If the strains are 'sufficiently' small, the equations are linear, and the relation between stress and strain is a kind of generalised Hooke's law.

Poisson and Navier based their work on a particular model of the microscopic structure of a solid: one in which particles are held together by mutual attractions. More satisfactory is the continuum model of a solid or fluid material, according to which the particle nature of matter is ignored. Material properties, such as density, are defined at a 'point' by taking the limit of the ratio of mass to volume as the volume shrinks to a size which still contains a large number of molecules. (For a fuller discussion see, for instance, Batchelor (1967).) We assume that this limit exists in an unambiguous sense. In a similar way, a 'material particle' will be defined to be a volume of material whose dimensions are small compared with all other relevant length scales, but which contains sufficient molecules for its precise microscopic structure to be irrelevant for considerations of dynamics. We may therefore use an ideal continuum as our mathematical model of a solid; one in which density may be defined as the ratio of mass to volume as the volume shrinks to *zero*, and a material particle is a geometrical point.

Since the 1940s, the description of stress and strain in continuous materials and of the relation between them has developed dramatically (see, for instance, Leigh 1968). Most of this work, however, has been related to the non-linear mechanics of materials in which strains are no longer small, and the influence on the linear theory has been slight, except to show it more clearly as a perturbation on the more general non-linear system. The discussion in this chapter

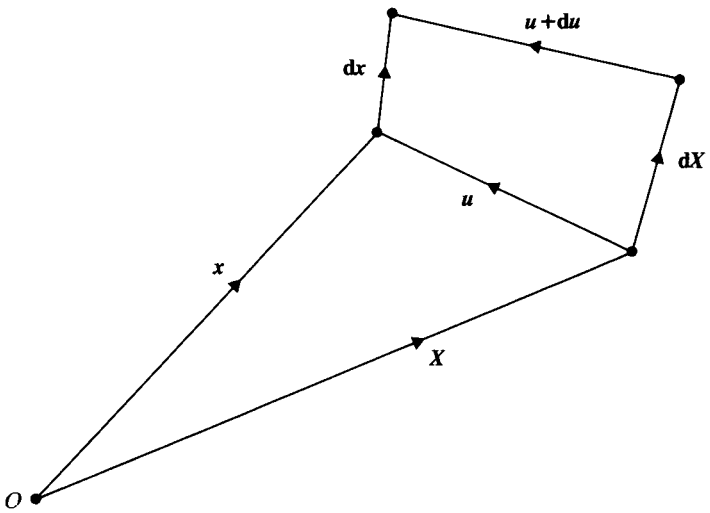

Fig. 1.1 Displacements of neighbouring material particles.

follows the classical argument. In addition, it is rather condensed, since there already exists a number of readily available accounts of a similar type (see, for instance, Sokolnikoff 1956). Here we shall emphasise those aspects which are particularly important from the point of view of the dynamical theory.

1.1 Deformation and strain

Suppose that, in the deformation of a material from an original reference state, a material particle moves from a point $\boldsymbol{X}$ to a point $\boldsymbol{x}$ at time t, where both $\boldsymbol{X}$ and $\boldsymbol{x}$ are referred to a fixed background frame; $\boldsymbol{X} \equiv (X_1, X_2, X_3)$, $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ in rectangular coordinates (see fig. 1.1). The change in material configuration due to the deformation can be considered as a one-to-one mapping and represented by the functional relation

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t); \quad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t).$$

It will be assumed that $\boldsymbol{x}(\boldsymbol{X}, t)$ and $\boldsymbol{X}(\boldsymbol{x}, t)$ are continuous and differentiable functions of their arguments. The differential interval between two neighbouring material points is

$$\mathrm{d}S = (\mathrm{d}X_i \mathrm{d}X_i)^{1/2}$$

in the reference state, and

$$ds = (dx_i dx_i)^{1/2}$$

in the deformed state.

If the distance between any two material points is the same in both the reference and the final states, no distortion has occurred; the final state can be reached by a rigid-body translation and rotation of the material from the reference state. Clearly, the material will be distorted in shape only if distances between particles are changed. The local distortion in the neighbourhood of a material point is characterised by changes in differential interval. Now,

$$ds^2 - dS^2 = dx_i dx_i - dX_i dX_i$$

$$= \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i dx_j$$

and the local distortion, or strain, is completely described by the tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right),$$

known as the Almansi strain tensor.

We can also write

$$ds^2 - dS^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i dX_j,$$

showing that we can equally well use the Green strain tensor,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right),$$

to describe the local strain.

The Almansi tensor is appropriate to an Eulerian formulation, where the x_i are taken as independent coordinates defining a field point, and the Green tensor to a Lagrangian formulation, where the X_i are independent coordinates defining a material point.

The displacement of a material particle from its original position is given by

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X},$$

and so the Almansi tensor may be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).$$

If the deformation gradients $\partial u_i/\partial x_j$ are everywhere small, ε_{ij} is equal, to first order in $\partial u_i/\partial x_j$, to

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.1}$$

which is the Cauchy strain tensor.

It also follows that

$$\frac{\partial}{\partial x_j} = \left(\delta_{jk} - \frac{\partial u_k}{\partial x_j}\right)\frac{\partial}{\partial X_k} = \frac{\partial}{\partial X_j}$$

to leading order, so that it is immaterial, to the first order of approximation, whether derivatives are calculated with respect to the coordinates X_i or x_i. The Lagrangian and Eulerian formulations lead in these circumstances to the same result. In particular, the Green tensor also approximates to the Cauchy tensor; to first order,

$$E_{ij} = e_{ij}.$$

The Cauchy strain tensor is a symmetric tensor and therefore may be diagonalised by an appropriate rotation of coordinates into the principal axes of strain. The corresponding diagonal terms are called the principal strains. When not in diagonal form, the off-diagonal terms ($e_{ij}, i \neq j$) are called shear strains.

Each elementary vector distance $\mathbf{d}\boldsymbol{X}$ between two material particles suffers a change in magnitude specified (when the deformation gradients are small) by the Cauchy strain tensor, as we have just seen. However, the e_{ij} are not sufficient to specify completely the change in direction of $\mathbf{d}\boldsymbol{X}$. The vector displacement of $\mathbf{d}\boldsymbol{X}$ into $\mathbf{d}\boldsymbol{x}$ is $\mathbf{d}\boldsymbol{u}$, and

$$\begin{aligned} \mathrm{d}u_i &= (\partial u_i/\partial x_j)\,\mathrm{d}x_j \\ &= e_{ij}\mathrm{d}x_j + \omega_{ij}\mathrm{d}x_j, \end{aligned}$$

where

$$\omega_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i). \tag{1.2}$$

We may write

$$\omega_{ij} = -\tfrac{1}{2}\varepsilon_{ijk}\omega_k,$$

where the rotation vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}. \tag{1.3}$$

Thus $\mathbf{d}\boldsymbol{u}$ is composed of a deformation term $\boldsymbol{e}\cdot\mathbf{d}\boldsymbol{x}$ together with a rotation by an amount $\frac{1}{2}\boldsymbol{\omega}\wedge\mathbf{d}\boldsymbol{x}$.

If ω is zero, then

$$du_i = e_{ij}dx_j,$$

and the deformation is one of 'pure strain'. If, for instance, $\mathbf{d}\boldsymbol{x}$ lies along a principal axis of strain, its displacement $\mathbf{d}\boldsymbol{u}$ lies in the same direction; the vector element is stretched or compressed but not rotated. If $\boldsymbol{e}$ is zero, then every element is rotated without change of length by the same amount, $\frac{1}{2}\boldsymbol{\omega}$. The material moves locally as a rigid body.

Since we are considering deformations in which both the strain and the rotation are small, the two operations commute, to the accuracy of the first order. (If the deformation gradients are not small, the deformation can still be described in terms of the operations of a pure strain and a rotation which, however, do not commute (see Leigh 1968).) For the same reason, when the deformation gradients are small, two separate deformations may be superimposed in any order and the net strain is the linear sum of the two separate strains.

If the displacement of a point is $\boldsymbol{u}(\boldsymbol{x},t)$, its velocity $\boldsymbol{v}(\boldsymbol{x},t)$ is given by

$$v_j(\delta_{ij} - \partial u_i/\partial x_j) = \partial u_i/\partial t,$$

and so

$$\boldsymbol{v} = \partial \boldsymbol{u}/\partial t = \dot{\boldsymbol{u}} \tag{1.4}$$

to the first order. Similarly, the acceleration is

$$\boldsymbol{a}(\boldsymbol{x},t) = \partial \boldsymbol{v}/\partial t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v},$$

and if $\boldsymbol{v} = \partial \boldsymbol{u}/\partial t$ is also small (compared with some characteristic velocity),

$$\boldsymbol{a} = \partial \boldsymbol{v}/\partial t = \partial^2 \boldsymbol{u}/\partial t^2 = \ddot{\boldsymbol{u}} \tag{1.5}$$

to the first order.

1.2 Continuity of mass

The density $\rho(\boldsymbol{x})$ at a point $\boldsymbol{x}$ is defined, as we have said, to be the limit of the ratio of the mass of a region of the material surrounding the point to its volume as the region of material shrinks to the point itself. It is assumed that this limit is unique.

The mass of a arbitrary body of material occupying volume V in

the deformed state and volume V_0 in the reference state is

$$M = \int_V \rho(\boldsymbol{x})\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

$$= \int_{V_0} \rho_0(\boldsymbol{X})\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3, \tag{1.6}$$

where ρ_0 is the density in the reference state.

If we change variables of integration in the first integral to X_i, the range of integration changes to V_0 and we get

$$M = \int_{V_0} \rho\{\boldsymbol{x}(\boldsymbol{X})\}\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3, \tag{1.7}$$

where $\partial(x_1,x_2,x_3)/\partial(X_1,X_2,X_3)$ denotes the Jacobian. Assuming that the density and deformation gradients are continuous, we obtain from equations (1.6) and (1.7)

$$\rho(\boldsymbol{x})\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} = \rho_0(\boldsymbol{X}),$$

since V_0 is an arbitrary volume.

If the deformation gradients $\partial u_i/\partial x_j$ are small,

$$\frac{\partial(X_1,X_2,X_3)}{\partial(x_1,x_2,x_3)} = 1 - \frac{\partial u_k}{\partial x_k}$$

to first order, and so

$$\rho(\boldsymbol{x}) = \rho_0(\boldsymbol{X})(1 - \partial u_k/\partial x_k) \tag{1.8}$$

to the first order.

The quantity,

$$\theta = \partial u_k/\partial x_k = e_{kk}, \tag{1.9}$$

is called the dilatation and is equal to the proportional increase in volume of a given mass of material. θ is the trace of the strain tensor $\boldsymbol{e}$ and is equal to the sum of the principal strains. In an incompressible material, ρ remains constant and θ is zero.

The strain tensor itself may be separated into a dilatational part and a deviatoric part:

$$e_{ij} = \tfrac{1}{3}\theta\delta_{ij} + \bar{e}_{ij}. \tag{1.10}$$

This equation defines the deviatoric strain $\bar{e}_{ij}$ which clearly has zero trace, and on its own represents an equivoluminal strain.

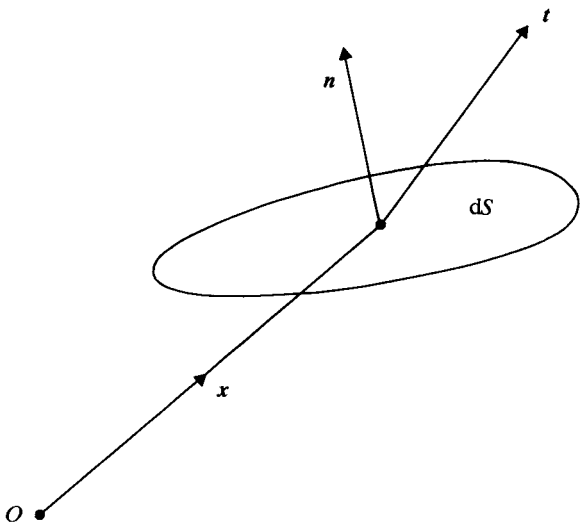

Fig. 1.2 Traction on an element of surface.

1.3 Stress

Consider a plane element of surface lying within a material body. It is assumed that the action of the material on one side of the surface, on the material on the other side, is equivalent to that of a force acting at the centroid of the surface together with a couple. It is further assumed that, as the surface shrinks to a point, the ratio of the force to the surface area tends to a unique limit, and that the ratio of the couple of the surface area tends to zero. These assumptions are justified simply by the success of the theory in accounting for the behaviour of elastic materials.†

Let the surface have normal $\boldsymbol{n}$, centroid $\boldsymbol{x}$, and differential area $\mathrm{d}S$ (see fig. 1.2). Then the action of the material on the side to which $\boldsymbol{n}$ points, on the material on the other side of the surface, may be represented to first order by a force

$$\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S$$

and zero couple; $\boldsymbol{t}$ is called the surface traction. Different tractions will be defined depending on whether $\boldsymbol{n}$ and $\mathrm{d}S$ refer to the reference

† A material in which the ratio of the surface couple to the surface area does not vanish in the limit is called multipolar (see, for instance, Jaunzemis 1967).

or the deformed state. However, if the deformation gradients are small the difference may be neglected for first-order accuracy.

The total force acting on the material within an arbitrary volume V of the body with surface S is

$$\int_V \boldsymbol{F}\rho \,\mathrm{d}V + \int_S \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S,$$

where $\boldsymbol{n}$ is the outward normal at each point of S, and $\boldsymbol{F}$ is a possible body force per unit mass (e.g. gravity, where $\boldsymbol{F}=\boldsymbol{g}$). The total moment of forces about the origin is

$$\int_V \boldsymbol{x}\wedge\boldsymbol{F}\rho \,\mathrm{d}V + \int_V \boldsymbol{M}\rho \,\mathrm{d}V + \int_S \boldsymbol{x}\wedge\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S,$$

where $\boldsymbol{M}$ is a possible body moment per unit mass (e.g. magnetic moment).

Euler's hypothesis for a continuum (analogous to the laws of particle mechanics) is that the total force on a body of material equals the rate of change of the total momentum of the body, and that the total moment of forces is equal to the rate of change of moment of momentum. That is, if υ and $\boldsymbol{a}$ are the local velocity and acceleration of the material,

$$\int_V \boldsymbol{F}\rho \,\mathrm{d}V + \int_S \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\upsilon \,\mathrm{d}V$$
$$= \int_V \rho\boldsymbol{a}\,\mathrm{d}V \tag{1.11}$$

$$\left(\text{since } \frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{q}\rho \,\mathrm{d}V = \int_V \frac{\partial \boldsymbol{q}}{\partial t}\rho \,\mathrm{d}V \text{ for all differentiable functions } \boldsymbol{q}\right)$$

and

$$\int_V \boldsymbol{x}\wedge\boldsymbol{F}\rho \,\mathrm{d}V + \int_V \boldsymbol{M}\rho \,\mathrm{d}V + \int_S \boldsymbol{x}\wedge\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{x}\wedge\upsilon\rho \,\mathrm{d}V$$
$$= \int_V \boldsymbol{x}\wedge\boldsymbol{a}\rho \,\mathrm{d}V \tag{1.12}$$

If we take V to be an arbitrary disc-shaped region with faces S^+ and S^- and small thickness ε (see fig. 1.3), the second term in equation (1.11) becomes

$$\int_{S^+} \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S + \int_{S^-} \boldsymbol{t}(-\boldsymbol{n},\boldsymbol{x},t)\mathrm{d}S + O(\varepsilon),$$

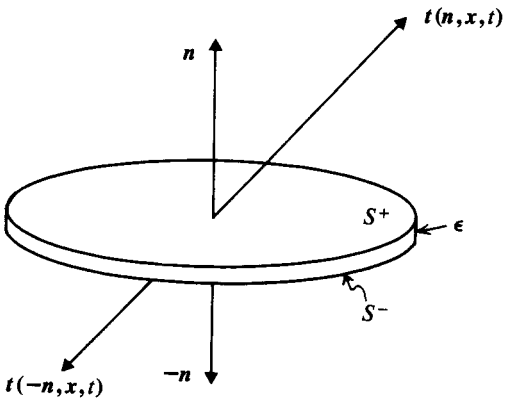

Fig. 1.3 Tractions on a thin disc-shaped region.

where $\boldsymbol{n}$ is the outward normal on S^+, while the other two terms are both $O(\varepsilon)$. We now let $\varepsilon \to 0$. Since S^+ is arbitrary we have

$$\boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t) = -\boldsymbol{t}(-\boldsymbol{n},\boldsymbol{x},t), \tag{1.13}$$

if $\boldsymbol{t}$ is a continuous function of $\boldsymbol{x}$ on S^+; i.e. the traction on the material on one side of an element of surface due to the material on the other side is equal and opposite to the traction imposed by the first on the second.

The traction on a surface with normal parallel to a coordinate axis ($\boldsymbol{n} = \boldsymbol{e}_i$) is

$$\boldsymbol{\sigma}_i = \boldsymbol{t}(\boldsymbol{e}_i,\boldsymbol{x},t) = (\sigma_{i1},\sigma_{i2},\sigma_{i3}).$$

The elements $\sigma_{ij}(\boldsymbol{x},t)$ form a 3×3 array. We now show that it is a tensor.

Consider a small volume of material in the shape of a tetrahedron with three faces normal to the coordinate directions and the fourth face with normal $\boldsymbol{n}$ and area ε^2 (see fig. 1.4). The areas of the first three faces are $n_1\varepsilon^2, n_2\varepsilon^2$ and $n_3\varepsilon^2$. The resultant of the tractions on the surface of the tetrahedron is

$$\varepsilon^2[-n_1\boldsymbol{t}(\boldsymbol{e}_1,\boldsymbol{x},t) - n_2\boldsymbol{t}(\boldsymbol{e}_2,\boldsymbol{x},t) - n_3\boldsymbol{t}(\boldsymbol{e}_3,\boldsymbol{x},t) + \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t)] + O(\varepsilon^3),$$

(where $\boldsymbol{x}$ is the position vector of any point within the tetrahedron) if $\boldsymbol{t}$ is continuous. If the body force and acceleration are bounded in magnitude, equation (1.11) gives

$$-n_1\boldsymbol{\sigma}_1(\boldsymbol{x},t) - n_2\boldsymbol{\sigma}_2(\boldsymbol{x},t) - n_3\boldsymbol{\sigma}_3(\boldsymbol{x},t) + \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x},t) = O(\varepsilon).$$

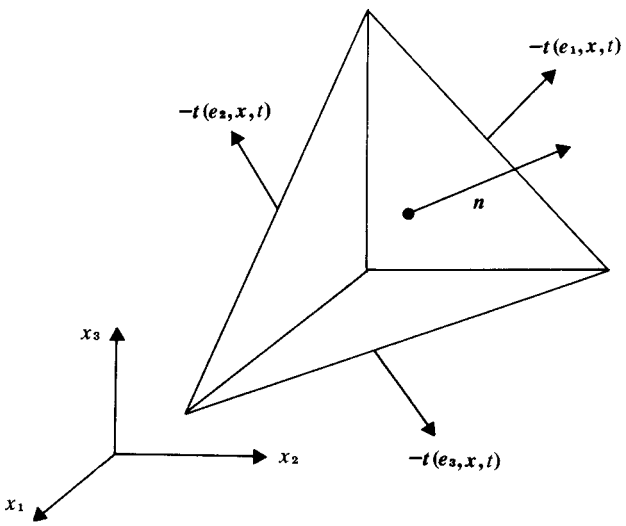

Fig. 1.4 Tractions on a tetrahedral-shaped region.

We now let $\varepsilon \to 0$. It follows that, at any point $\boldsymbol{x}$ of the material,

$$t_i(\boldsymbol{n},\boldsymbol{x},t) = n_j \sigma_{ij}(\boldsymbol{x},t). \tag{1.14}$$

Since $\boldsymbol{t}$ and $\boldsymbol{n}$ are vectors, and $\boldsymbol{n}$ is arbitrary in direction, $\boldsymbol{\sigma}$ must be a tensor.

If $\boldsymbol{t}$ is originally defined with normal and surface area referring to the material in the current (deformed) state, $\boldsymbol{\sigma}$ is called the Cauchy stress. If $\boldsymbol{n}$ and $\mathrm{d}S$ refer to the original reference state, $\boldsymbol{\sigma}$ is the 'nominal stress'. However, as noted above, the two tensors are equal to first order when the deformation gradients are small.

1.4 The momentum equations and their consequences

The equation of linear momentum (equation (1.11)) can be rewritten with the help of equation (1.14) to give

$$\int_V (F_i - a_i)\rho \,\mathrm{d}V + \int_S \sigma_{ij} n_j \,\mathrm{d}S = 0.$$

Transformed by use of the divergence theorem, this becomes

$$\int_V [\partial_j \sigma_{ij} + \rho(F_i - a_i)]\,\mathrm{d}V = 0,$$