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Stopping time techniques for analysts and probabilists

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PREFACE

Adapted sequences of integrable functions arise naturally in probability theory. Martingales, submartingales and supermartingales especially are very important to probabilists since they serve as mathematical models for many probabilistic phenomena. Consider for instance the fortune of a gambler. The martingale condition corresponds to the situation where this fortune remains constant in the sense of conditional mean. The supermartingale condition corresponds to the situation where at each play the game is unfavorable to the gambler in the same sense, while the submartingale condition corresponds to the situation where at each play the game is favorable in that sense. It is therefore clear that these notions are extremely important in probability theory, and so they have been heavily studied. One of the most interesting questions is when (and to what) does such an adapted sequence converge almost everywhere?

Such classes of adapted sequences do not only have interest in probability theory. They have also been used in other branches of mathematics such as potential theory, dynamical systems and many others.

However it is my feeling that not many analysts are used to dealing with martingales. That is even more the case with extensions of the martingale notion, involving stopping times. Nevertheless stopping time techniques do have many applications in real or functional analysis. This is what this book is about : to be of use to probabilists (of course) but also to analysts, by introducing them to the most important stopping time techniques. To look at a problem in analysis (real or functional) bearing in mind stopping time results may illuminate it and sometimes yield a surprisingly simple and elegant solution. As an example in real analysis there is the easy proof of the Radon-Nikodym theorem using stopping time techniques, given by Edgar and Sucheston. Also a surprising relationship between convergence in probability (i.e. in measure) and almost everywhere convergence can be described using stopping time techniques.

Furthermore, stopping times are very important in the study of topological or geometrical properties in Banach spaces. Many of them can be characterized by using stopping times. For a short description of the most important applications, see further on in this preface.

It is this double importance of the theory – for probabilists as well as for analysts – that encouraged me to study the convergence theory of generalized martingales in Banach spaces (including many new results in $\mathbb{R}$, the real line) and of extensions of generalized sub- or supermartingales in Banach lattices (also including many new results in $\mathbb{R}$).

In this book only some familiarity with measure theory as well as with functional analysis is presupposed. Concerning convergence of adapted sequences we only presuppose knowledge of the following theorem, to be found in Neveu [1975], theorem IV-1-2, p.62-64 and theorem II-2-9, p.26-29.

Theorem (J.Doob) : Let $(X_n, F_n)_{n \in \mathbb{N}}$ be a submartingale such that

$$\sup_{n \in \mathbb{N}} \int X_n^+ < \infty$$

then there exists $X_\infty \in L^1$ such that

$$\lim_{n \to \infty} X_n = X_\infty, \text{ a.e. } .$$

In this theorem obviously the following results are included :

a) Every real L^1-bounded martingale converges a.e. to a function $X_\infty \in L^1$.
b) Every positive supermartingale converges a.e. to a function $X_\infty \in L^1$ (but it must be emphasized that in fact (b) is used in the proof of the theorem above).

For new proofs of the real martingale convergence theorem, see Isaac [1965] or Lamb [1973].

Strictly speaking the above results and their proofs appear in this book independently : see theorem II.2.4.3, theorem III.2.2, theorem VII.2.12, theorem VIII.1.7, remark VIII.3.1, which yield several independent proofs of the results above, but this is more a curiosity and is not exactly the way in which to introduce Doob's result in classical probability courses.

So, apart from some elementary facts, the present book is completely self-contained.

Throughout the book E will stand for a Banach space with dual E' and (Ω,F,P) will be a complete probability space.

Chapter I repeats some basic notions which are constantly used throughout the book. Classical measure theoretic notions and results are mentioned : we have first scalar and strong measurability and their relation expressed in the theorem of Pettis. Concerning integrability, the Bochner integral is introduced and the spaces L^p_E $(p \in [1,+\infty])$. On L^1_E not only is the $\|\cdot\|_1$-norm considered but also the Pettis-norm of the form, where $f \in L^1_E$

$$\|f\|_{Pe} = \sup_{\substack{x' \in E' \\ \|x'\| \leq 1}} \int_\Omega |x'(f)|dP$$

Relations between $\|\cdot\|_1$ and $\|\cdot\|_{Pe}$ are given with a quick proof. Then we give the definition of an adapted sequence, which is the real subject of the book. By an adapted sequence we mean a sequence $(X_n,F_n)_{n\in\mathbb{N}}$, where $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of sub-σ-algebras of the σ-algebra F and where $X_n \in L^1_E$ for every $n \in \mathbb{N}$, such that every X_n is F_n-measurable. For these processes we define the notion of stopping times, and for bounded stopping times, the elementary properties are derived, such as f.i. the "localization property". The existence of conditional expectations of a function $f \in L^1_E$ w.r.t. a sub-σ-algebra G of F, i.e. a function $E^G f$ such that

$$\int_A E^G f = \int_A f$$

for every $A \in G$, is proved. Also some elementary properties of E^G are indicated. Much of our attention in this book is focused on the problem of convergence of adapted sequences. Therefore, different types of convergence are discussed : pointwise almost everywhere (a.e.), $\|\cdot\|_1$-convergence, $\|\cdot\|_{Pe}$-convergence, convergence in probability. This last notion is studied for $(X_n)_{n\in\mathbb{N}}$ as well as for $(X_\tau)_{\tau\in T}$, where τ runs through the set of bounded stopping times and where

$$X_\tau = \sum X_n \chi_{\{\tau=n\}} \quad ;$$

here the sum ranges over all possible values τ can take (there are only finitely many). We have here the important result of Millet-Sucheston, which says that if $(X_\tau)_{\tau \in T}$ converges in probability, then it converges a.e., and hence also $(X_n)_{n \in \mathbb{N}}$. This elementary but important result was only proved in 1979. In 1980, Bellow-Egghe gave a simpler proof for it (including a small correction). It is this proof that is presented here; the method is important for what follows later on in the book. We feel that this result belongs in every elementary course on probability and perhaps also on analysis.

We have given a rather elaborate description of the introductory chapter I, to make this book more accessible for the non-expert, to whom this book is certainly directed. But I also hope that the expert will find the book useful since it unifies the material of numerous articles, selects the most elegant proofs, as we shall indicate later on, and since it contains a lot of very recent results. The book can be used as a seminar text in probability as well as in functional analysis. It is also hoped that it will encourage analysts to consider stopping time techniques occasionally and also that probabilists will understand the power of functional analysis and more specifically of Banach spaces, in probability theory. Let us now continue to describe the contents of the book.

Chapter II proves martingale convergence theorems in the framework of Banach spaces with the Radon-Nikodym property (RNP). The maximal lemma is very important in this connection as well as later on. It is however simple to prove. The special relation of martingale convergence theory and the geometry of Banach spaces is illustrated several times and, more specifically, the geometric notion of "dentability" in a Banach space is proved to be equivalent with the martingale convergence theorem : theorem of Rieffel, Huff et al.. The central result of this chapter is the a.e. convergence of martingales in Banach spaces with (RNP), known usually as Chatterji's result. In our book we call it the result of A. and C. Ionescu-Tulcea since they were the first to prove it. However in our book the proof of Chatterji is presented since it is short, elegant and independent of the real martingale convergence theorem of Doob. Two other proofs of this fundamental theorem are presented. One is based on the Kadec-Klee renorming theorem of Banach spaces which is, roughly speaking, a method of reducing the vectorvalued convergence problem to the real, already known, case. The other proof, that of Chacon-Sucheston, uses an optimal stopping method by which the problem is reduced to that of martingales of the form $(E^{F_n}X, F_n)_{n \in \mathbb{N}}$, the convergence of which is fairly easy to prove. All these proofs are given because of the importance of the various methods for what follows in the book. The chapter closes with some martingale convergence properties in general Banach spaces, possibly without (RNP) : results of Korzeniowski, Chatterji and Burkholder-Shintani. It is my feeling that these results are not so well-known to the mathematical public and since the results and their proofs are nice, they are interesting enough to be included in this book. Especially the result of Korzeniowski-relating martingale convergence with weak convergence of the associated sequence of probability distributions-uses results a little beyond the scope of this book; f.i. the measurable selection theorem of Kuratowski and Ryll-Nardzewski is used in the proof.

Chapter III contains proofs of convergence theorems for sub- and supermartingales with values in Banach lattices. We prove first the theorem of Heinich on convergences a.e. of positive submartingales, with the original proof as well as the proof by using the renorming theorem of Davis-Ghoussoub-Lindenstrauss, in the same way as in the previous chapter the Kadec-Klee renorming theorem was used. Then the results of Szulga-Woyczynski on convergence of general submartingales are presented, with some new proofs. Also the result of Benyamini-Ghoussoub on the convergence of supermartingales is included. A result of Ghoussoub-Talagrand on weak convergence a.e. of class (B) supermartingales is postponed until chapter V, since for the proof more machinery is needed, which is available in chapter V. Also this chapter closes with a submartingale convergence theorem of Davis-Ghoussoub-Lindenstrauss, with values in general Banach spaces.

Chapter IV contains the basic inequalities - proved by Bellow-Egghe - upon which chapter V and VII are based to yield extensions of the convergence results of chapter II. It describes the validity of inequalities of the following form, where $(X_n, F_n)_{n \in \mathbb{N}}$ is an adapted sequence with values in a Banach space :

$$\limsup_{m,n \in \mathbb{N}} \|X_n(\omega) - X_m(\omega)\| \leq C. \limsup_{\substack{n,m \in \mathbb{N} \\ n \geq m}} \|E^{F_m} X_n(\omega) - X_m(\omega)\|$$

valid a.e..The norm can also be replaced by a continuous seminorm. Also stopping times instead of natural numbers can be involved. So if the righthand side is zero, i.e. a generalization of the martingale concept, then the left hand side is zero, i.e. convergent a.e.. The validity of such inequalities is studied and, as can be remarked, the results are very sharp : first they yield in chapter V and VII all known convergence results of extensions of martingales, thus giving a new unified proof for them, and secondly it is shown by Edgar and Mc Cartney-O'Brien that the conditions which were needed in the proof of these inequalities are really needed : the above inequality fails in certain (RNP) spaces which do not embed into a separable dual.

Chapter V contains the theory of uniform amarts, amarts, weak sequential amarts, weak amarts and (uniform) semiamarts. The class of uniform amarts comprises the class of quasi-martingales, yet the martingale convergence theorem of A. and C. Ionescu-Tulcea remains true for uniform amarts without additional assumptions. This was proved by Bellow, but becomes trivial using the Bellow-Egghe inequalities in chapter IV. Quasi-martingales arise in a natural way in the geometric theory of Banach spaces. However uniform amarts are easier to study than quasi-martingales. An application of the uniform amart a.e. convergence theorem to the geometric theory of Banach spaces is given. Next amarts - a contraction of the term asymptotic martingales - are studied. It is proved that convergence in the $\|\cdot\|_{Pe}$-norm obtains for uniformly integrable amarts in Banach spaces with (RNP), which is a result of Uhl but which is proved here by using again the Bellow-Egghe inequalities of chapter IV. Using amarts, a probabilistic characterization of the spaces E for which every operator from L^1 into E has a compact restriction to L^∞, is given by Egghe, based on a result of Uhl and another of Diestel. For applications of amarts : see chapter VI. Weak convergence a.e. is obtained for weak sequential amarts of class (B) if and only if E and E' have (RNP), again using the Bellow-Egghe inequalities of chapter IV. This is a result of Brunel-Sucheston. Another result of Brunel-Sucheston, giving a characterization of the separability of the dual E' in terms of a class of weak sequential amarts, is given by presenting the new proof of Edgar, which also gives an extension of this result, by characterizing Asplund operators using a class of weak sequential amarts. In chapter V the result of Ghoussoub-Talagrand, mentioned in chapter III, is also proved. Next a characterization of reflexivity in terms of weak convergence of weak amarts is given, being another result of Brunel-Sucheston. A study of (uniform) semiamarts concludes this chapter. These arise naturally in chapter IV, showing their importance in probability theory as well as in the geometric theory of Banach spaces. In chapter V we give some oscillatory results on semiamarts, using the notion of Banach limit.

Chapter VI deals with some applications of the amart convergence theory, as developed in chapter V. They were not included in chapter V since the applications are in the field of adapted nets, indexed by $Z \times Z$ or even more generally. By making a separate chapter, we can introduce these notions properly, and the results are separated from the general theory. Also, chapter VI can be skipped by the reader who is not interested in these topics, without losing continuity. We present some results of Millet-Sucheston and another of Astbury. In this last result an old problem of Krickeberg concerning the characterization of the Vitali condition for $(F_i)_{i \in I}$ in terms of convergence of adapted nets $(X_i, F_i)_{i \in I}$ is solved, using amarts (as was shown previously by Millet-Sucheston, martingales can not be used here!). In the notes and remarks of chapter VI, we also reveal a didactical application of amarts, mentioned by Edgar-Sucheston.

Chapter VII continues chapter V and yields another bunch of theorems which are applications of the inequalities proved in chapter IV. The chapter opens with some negative results on vector amart convergence theory. It shows that amarts never converge strongly a.e., except in the trivial case when E is finite dimensional (results of Bellow with proofs of Egghe). New extensions of the martingale concept are defined : pramarts, mils, and it is shown that they have better strong convergence properties than amarts (results of Bellow-Dvoretzky, Millet-Sucheston). Also, if $E = \mathbb{R}$, they generalise amarts and Doob's theorem extends to these new concepts without additional hypothesis (theorem of Mucci).

Chapter VIII studies extensions of sub- and supermartingales : subpramarts and superpramarts. For $E = \mathbb{R}$ it is shown that the sub- and supermartingales a.e. convergence theorems extend to sub- and superpramarts (Millet-Sucheston). The theorem of Heinich in chapter III extends to subpramarts without any additional hypothesis. This is a result of Słaby, solving a problem of Egghe. A second proof of Frangos is also presented, based on Talagrand's theorem that a Banach lattice with (RNP) is a separable dual. Both proofs of Słaby and Frangos are very recent and based on lemmas of Egghe. This result is applied by Słaby to yield a partial result on Sucheston's problem on pramartconvergence in Banach spaces. Frangos did the same, obtaining another partial result on the same problem. We now have that in a Banach space E with (RNP), every pramart with an L^1_E-bounded subsequence converges strongly a.e. provided that E is weakly sequentially complete (Słaby) or if E is isomorphic with a subspace of a separable dual (Frangos). Positive superpramarts converge a.e. in a Banach lattice only in the case when E is a sublattice of $\ell^1(\Gamma)$ (Egghe). Finally a mean convergence result of Egghe is proved for games which become worse (better) with time, extending a result of Subramanian, without actually using it.

Chapter IX closes the study developed in this book by making some remarks. First scalar convergence results are proved for the earlier introduced types of adapted sequences. Then a summary of the most important convergence results proved in this book is given. Finally the concept of Pettis integrable function - extending the notion of Bochner integrable function - is defined and some properties are mentioned. Also some results and literature on convergence of adapted sequences of Pettis integrable functions are mentioned, indicating that there is still a lot which has to be done in this area.