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Localization in Noetherian Rings

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PREFACE

The theory of Noetherian rings may be said to have begun with Goldie's 1958 paper. Since then the theory has steadily gathered strength. By now, various methods and results from the theory of commutative Noetherian rings have been adapted to non-commutative Noetherian rings. The theory has also contributed to and benefitted from developments in three nearby areas: P.I. rings, enveloping algebras of Lie algebras, and group rings of polycyclic-by-finite groups.

A major problem still confronting the theory of Noetherian rings concerns localization. To get an idea of this problem, recall that several fundamental results in the theory of commutative Noetherian rings are obtained by using the procedure of localization at prime ideals. It is then natural to expect - at least to hope - that a similar procedure which enables one to localize Noetherian rings at prime ideals would have a similar salutary effect on the theory of Noetherian rings. The problem, in a nutshell, is to find such a procedure.

During the last decade and half, quite a bit of effort has been made to find such a procedure. By now, a certain procedure has emerged as the 'correct' one. This standard procedure takes the commutative situation and the situation dealt with in Goldie's Theorem (2.3.7) as models, and attempts to use Ore's method of localization to localize Noetherian rings at semi-prime ideals. The general belief as to the 'correctness' of the standard procedure is based on some internal evidence along with some guesswork. Be that as it may, since the standard procedure works only on the so-called classically localizable semi-prime ideals, such ideals have come to be regarded as the basic objects in the study of localization in Noetherian rings.

When applied to classically localizable semi-prime ideals, the standard procedure works as well as may be expected; therein lies its strength. However, the standard procedure does not apply to all semi-prime ideals; and when it fails to apply, it does so with no explanation and, often, for no apparent reason. Therein lies its weakness.

Until recently, the implications of this weakness of the standard procedure went almost unnoticed. The focus of the study of localization in Noetherian rings was the recognition problem: how does one recognize whether a given semi-prime ideal is classically localizable? As an answer, a variety of necessary and/or sufficient conditions were developed. Many of these were meant to be used in special types of rings (e.g. HNP rings) or on special types of ideals (e.g. the nil radical). For an obscure reason, a great deal of attention was paid to Noetherian orders in Artinian rings; these are Noetherian rings in which the nil radical turns out to be classically localizable in a particularly nice way. All this is fine so far as it goes and quite a bit of it is of interest in its own right. However, since it is focused elsewhere, it is of no help in dealing with semi-prime ideals which fail to be classically localizable.

The gravity of the aforementioned weakness of the standard procedure recently became clear from studies of enveloping algebras of solvable Lie algebras and of group algebras of polycyclic-by-finite groups. There is plenty of evidence available now showing that, as non-commutative Noetherian rings go, these rings are quite decent. Besides, these rings have come to be regarded as among the premier examples of situations in which the theory of Noetherian rings can find its natural 'applications'. However, the standard procedure of localization does not work well in these rings since these rings often fail to have enough classically localizable semi-prime ideals.

All these factors have led to an apparently no-win situation - a stymie - in the study of localization in Noetherian rings: on the one hand, the standard procedure of localization appears to be the only 'correct' one possible. On the other hand, the standard procedure works only when there are enough classically localizable semi-prime ideals available; and, at least in certain premier classes of decent Noetherian rings, enough classically localizable semi-prime ideals are often unavailable.

The distressing lack of applications of localization to study other aspects of Noetherian rings is but a reflection of this stymie. Now, it is patent that the study of localization cannot be an end in itself. The point in developing machinery of localization has to be its usefulness in doing something else (and, hopefully, non-trivial) at least for the well-known classes of decent Noetherian rings. That being so, one faces two alternatives. The first one is to admit that nothing much can be done to salvage the situation and let it go at that. In that case, one would have to learn how to develop most, if not all, of the theory of Noetherian rings without localization. Some recent research seems to be directed to just that end. The other alternative is to come up with a new point of view about localization, show that it complements the old point of view, and show that it is not plagued with the shortcomings of the old point of view; and if one could offer some convincing reasons to believe that the new point of view would have some genuine 'applications', so much the better.

The central theme of this monograph is to show that the latter alternative is not infeasible. A change in attitude to localization is called for and a lot more work is needed to clarify the shape and size of the undertaking suggested here. However, the point of view suggested in this monograph appears to be a step in the right direction. Indeed, an earlier privately circulated version of the monograph has already led to some promising developments along the lines suggested here; cf., sections 6.2 and 7.2.

The view of localization suggested here hinges on two concepts: the second layer of indecomposable injectives and the links between prime ideals. The second layer of an indecomposable injective contains some crucial information about the structure of that injective. This concept is also related to 'primary decomposition' of modules. The concept of links between prime ideals is a subtlety of the prime spectrum and is related to the ideal structure of a given ring.

The concepts of the second layer and the links are introduced in chapters 4 and 5, respectively. The interaction between these concepts is studied in chapter 6, and is used in chapter 7 to study localization. The outcome has threefold significance: first, it brings out the 'real' reason for the stymie mentioned earlier. Secondly, it shows that, appearances notwithstanding, the study of localization is inextricably tied to the study of the structure of indecomposable injectives; so much so that one can make a strong case in favour of regarding the study of the structure of indecomposable injectives as an end in itself, with localization as but a means to this end. Thirdly, it shows that, under some apparently mild conditions, a nice procedure of localization is available in an important and large class of Noetherian rings. This class - the class of Noetherian rings satisfying the second layer condition - is studied further in chapters 8 and 9. The Appendix provides more information about four important types of Noetherian rings satisfying the second layer condition: HNP rings with enough invertibles, Noetherian P.I. rings, enveloping algebras of solvable Lie algebras, and group algebras of polycyclic-by-finite groups. The first three chapters provide the background needed for following the rest of the monograph.

Some broad directions for further research are discussed in the latter part of the monograph. Several open problems are also indicated. A few of them are well-known old problems. The rest are new; by and large, these problems are untested; however, not all of them appear unreasonably difficult and some of them may even be easy to solve.

It should be noted that, quite often, the proofs given in the text have little resemblance to those in the references cited. The citations in the text are mostly for the purpose of assigning credit for major ideas and results. An effort has been made to be accurate in this regard. The author-date system is used for citations, with P and U in place of the date for pre-prints and unpublished works, respectively. The reference list at the end of the monograph contains, for the most part, works that are actually cited in the text. That list should not be misconstrued as a documentation of all the works on the topics covered in the monograph. A complete account of the literature on Noetherian rings, up to 1979, can be found in appropriate sections of Reviews in Ring Theory, edited by L.W. Small (81).

It may be appropriate to mention that the monograph contains quite a bit of hitherto unpublished work of the present author. An outline of a substantial part of this unpublished work has been in circulation in the form of an announcement: Jategaonkar (79). Initially, the author's plan was to publish the details of the results in this announcement as a trilogy of papers. By now, two papers, Jategaonkar (81b), (82a), from the planned trilogy are in print. The third one, Jategaonkar (80U), was privately circulated but has remained unpublished. The first draft of this monograph, with this unpublished paper as its nucleus, was completed in early 1982 at the University of Warwick. A revised version was privately circulated in October 1982; an addendum was circulated in March 1983. The present version is an expansion and a reorganization of the 1983 version. It completes the planned trilogy.

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It is a pleasure to acknowledge the help and the active participation of Vasanti Jategaonkar, my wife, in all stages of the preparation of this monograph.

TO

LALITA

TERMINOLOGY AND NOTATION

All rings are non-zero, unitary, and associative; unless specifically stated otherwise, commutativity is not assumed.

All ring homomorphisms, subrings, and modules are unitary. All two-sided ideals are proper.

A ring R is a *prime ring* if the product of any two non-zero ideals of R is non-zero. A ring R is a *semi-prime ring* if R has no non-zero nilpotent ideals.

An ideal I of a ring R is a *prime ideal* resp. *semi-prime ideal* if R/I is a prime ring resp. a semi-prime ring.

The set of all prime ideals of a ring R is called the *prime spectrum* of R and is denoted as spec R.

The category of all right resp. left modules over a ring R is denoted as mod-R resp. R-mod. Module homomorphisms are written on the opposite side of scalars. For any $M \in$ mod-R, End M_R denotes the R-endomorphism ring of M, and $E(M_R)$ denotes the R-injective hull of M.

The definitions of all one-sided concepts are given for the right side only. A one-sided concept without an indication of the side to which it applies is assumed to apply to both sides. For instance, a *right Noetherian ring* is defined as a ring satisfying the ascending chain condition on the right ideals. It is understood that a left Noetherian ring is defined analogously; it is also understood that a *Noetherian ring* means a ring which is right Noetherian as well as left Noetherian.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, denote respectively the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers.

A possibly improper inclusion of sets is denoted by the sign $\subseteq$. The signs $\subset$ and $\subsetneq$ are reserved for strict inclusion. Thus, $X \subset Y$ means that X is contained in Y but is not equal to Y.

The complement of a set X in a set Y is denoted as $Y \setminus X$.

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The cardinality of a set X is denoted as $|X|$. A set X is *countable* if it is bijective with a subset of $\mathbb{Z}$.

The first infinite ordinal is denoted as ω.

References in round brackets such as (a.b.c) refer to the result c in section b of chapter a. The symbol $\square$ signifies the end of a proof. The same symbol placed at the end of a statement signifies that the proof is trivial and is omitted.

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