1 ORE'S METHOD OF LOCALIZATION

This chapter is the first of three introductory chapters. These chapters provide the background needed for following the rest of the monograph. For the most part, the material presented in these three chapters is an organization of fairly well-known concepts and results, although the specialist may find a few novelties in the manner in which some of the material is presented.

As stated in the preface, during the last decade and a half, a standard procedure of localization in Noetherian rings has emerged. This standard procedure is based on Ore's method of localization. The first two sections of this chapter are devoted to an examination of Ore's method in its full generality without any reference to the standard procedure of localization. Chapter 2 is devoted to Goldie's Theorem - a highly non-trivial illustration of the usefulness of Ore's method. The standard procedure of localization in Noetherian rings is modelled after the usual procedure of localization in commutative rings and after the way Ore's method is used in Goldie's Theorem. The standard procedure is introduced and studied in chapter 3. Ore's method is used, once again, in chapter 7 to develop a new procedure of localization in Noetherian rings.

Throughout this monograph, we shall primarily be concerned with Noetherian rings which badly fail to be commutative. Experience shows that any straightforward attempt to apply usual commutative methods to these rings leads nowhere. Several fairly common features of these rings seem anomalous in comparison with the features of commutative Noetherian rings.

In the third section of this chapter, several specific examples are supplied to facilitate acquaintance with Noetherian rings which badly fail to be commutative. Later in the monograph, these examples are repeatedly used for the purpose of illustration. Some of these examples are selected to illustrate the difficulties in employing Ore's method in a straightforward manner in developing machinery of localization in Noetherian rings.

1.1 QUOTIENT RINGS

Ore's method of localization is a transparent adaptation of the usual method of localizing a commutative ring at a multiplicative set.

In this section, we describe Ore's method and characterize the situation in which it is applicable. Some pleasant features of Ore's method are established in the next section.

Quotient rings: definition

Let R be a ring. By a *multiplicative set* D in R we mean a subset D of R which is closed under the multiplication in R, contains 1, but does not contain 0. Ore's method concerns the construction of a 'quotient ring of R relative to a multiplicative set D' in the sense set forth below.

Let D be a multiplicative set in R. A *right quotient ring of R relative to D* is an ordered pair (Q,f), where Q is a ring, and $f : R \to Q$ is a ring homomorphism satisfying the following three conditions:

(1) For every $d \in D$, $f(d)$ is a unit in Q;

(2) For every $q \in Q$, there exist $r \in R$ and $d \in D$ such that

$$q = [f(r)][f(d)]^{-1};$$

(3) $\ker f = \{r \in R \mid rd = 0 \text{ for some } d \in D\}$.

If (Q,f) is a right quotient ring as well as a left quotient ring of R relative to D then it is called a *two-sided quotient ring of R relative to D* or, more loosely, a *quotient ring of R relative to D*.

In dealing with these notions, it is customary to suppress the mention of f and to refer to Q as Ore's (right) quotient ring of R relative to D or as a *(right) Ore localization of R at D*.

The definition of a (right) quotient ring of R relative to D states the desired outcome of the (right) 'localization' of R at D. It does not provide an assurance that the desired ring exists (and for a good reason).

When D is a central multiplicative subset in R, an easy modification of the usual method suffices to construct a quotient ring of R relative to D. On the other hand, if R is the free algebra on at least two generators over a field and if D is the multiplicative set $R\backslash(0)$ then R does not have a (right) quotient ring relative to D; the simplest way to see this is to apply (1.1.1).

We mention that, as a rule, the rings with which we shall be concerned are Noetherian but badly non-commutative, and the multiplicative sets of interest to us are seldom central. In such situations, the existence of a (right) quotient ring is not always easy to figure out.

Quotient rings: existence and uniqueness

Our immediate goal is Ore's Theorem (1.1.1). This theorem provides some help in tackling the problem of verifying the existence of a (right) quotient ring of R relative to D by restating the problem in terms of certain internal conditions on D. These internal conditions are defined below.

A multiplicative set D in a ring R is called a *right Ore set* in R if it satisfies the *right Ore condition*: given $r \in R$ and $d \in D$, there exist $r_1 \in R$ and $d_1 \in D$ such that $rd_1 = dr_1$. A multiplicative set in R is called an *Ore set* in R if it is a right Ore set and a left Ore set in R.

A central multiplicative set in R is obviously an Ore set in R. Other examples of Ore sets are given a little later; cf., (1.1.4).

We observe that, under a surjective ring homomorphism $f : R \to \overline{R}$, a (right) Ore set D in R is mapped onto a (right) Ore set in $\overline{R}$, provided $D \cap \ker f = \emptyset$. Another useful observation is the *right common multiple property*: If D is a right Ore set in R then, given finitely many $d_1,\dots,d_n \in D$, there exist $d \in D$ and $r_1,\dots,r_n \in R$ such that $d = d_1r_1 = \dots = d_nr_n$. We omit the easy inductive proof of this property.

A multiplicative set D in a ring R is said to be *right reversible* in R if, for any $d \in D$ and $r \in R$ with $dr = 0$, there exists $d' \in D$ with $rd' = 0$. A multiplicative set in R is *reversible* in R if it is right reversible and left reversible in R.

We recall that an element d in a ring R is called a *right regular element* in R if $dr \neq 0$ for all non-zero $r \in R$. An element of R is called a *regular element* of R if it is right regular and left regular in R.

A multiplicative set D in R is said to be *(right) regular* in R if every element of D is (right) regular in R.

Now we prove

(1.1.1) *Ore's Theorem.* Let D be a multiplicative set in a ring R.

A (right) quotient ring of R relative to D exists iff D is a (right) reversible (right) Ore set in R.

If a (right) quotient ring of R relative to D exists then it is uniquely determined up to a natural isomorphism.

Proof. We tackle the one-sided case first and then observe that the two-sided case follows from it readily.

Existence: Assume that a right quotient ring (Q,f) of R relative to D exists. Observe that the definition of (Q,f) consistently interprets the 'quotient' of two elements as a 'right quotient'. Thus, for any $r \in R$ and $d \in D$, the 'left quotient' $[f(d)]^{-1}[f(r)]$ must be expressible as a 'right quotient' of the form $[f(r_1)][f(d_1)]^{-1}$ for some $r_1 \in R$ and $d_1 \in D$. Thus, $f(rd_1 - dr_1) = 0$. Using the specification of $\ker f$ in the definition of (Q,f), it follows that D is a right Ore set in R. The specification of $\ker f$ readily yields the right reversibility of D in R. This proves the 'only if' part.

Conversely, let D be a right reversible right Ore set in R. As in the commutative situation, we clear off the zero-divisors of elements of D first. To this end, set

$$\rho_D(R) = \{r \in R \mid rd = 0 \text{ for some } d \in D\}.$$

Using the right Ore condition on D and the right common multiple property of D, it is immediate that $\rho_D(R)$ is a two-sided ideal of R. Set $\bar{R} = R/\rho_D(R)$, and let $\bar{D}$ denote the image of D in $\bar{R}$. Then $\bar{D}$ is a right Ore set in $\bar{R}$. Moreover, using the right reversibility of D in R, it is easy to verify that $\bar{D}$ is a regular set in $\bar{R}$. Thus, after replacing R by $\bar{R}$ if necessary, we may assume that D is a regular right Ore set in R.

It remains to construct an over-ring Q of R such that every element of D is a unit in Q and such that every element of Q has the form rd^{-1} for some $r \in R$ and $d \in D$. As in the commutative case, this can be done by using the equivalence classes of an appropriate equivalence relation on $R \times D$. For the elaborate, but unedifying, details of this construction, we refer to Dixmier (77), or Jacobson (64), or Jategaonkar (70).

We provide another method of obtaining the required ring Q.

Let E_R be the R-injective hull of R_R, and let $S = \text{End } E_R$, the R-endomorphism ring of E_R. Set

$$I = \{s \in S \mid s(R) = 0\}, \quad \text{and} \quad T = \{s \in S \mid Is \subseteq I\}.$$

Clearly, T is a subring of S. Moreover, even though I may be just a left ideal of S, it is, in fact, a two-sided ideal of T. We plan to realize Q as a subring of T/I.

We observe first that there exists a natural ring embedding $g : R \to T/I$. For, given any $r \in R$, the left multiplication by r defines an R-endomorphism of R_R; this endomorphism can be extended to some R-endomorphism of E_R, say s_r. Clearly, $s_r \in T$. Moreover, although s_r need not be uniquely determined by r, it is obvious that $[s_r + I]$ is uniquely determined by r. The map $r \mapsto [s_r + I]$ provides the needed ring embedding $g : R \to T/I$.

We observe next that, for any $d \in D$, $g(d)$ is a unit in T/I. For, consider the map $s_d : E \to E$. Since R_R is an essential submodule of E_R and since d is a regular element of R, it follows that $\ker s_d = 0$. This makes $s_d(E)$ an injective submodule of E_R. Thus, $E = F \oplus s_d(E)$ for some R-submodule F of E. Now, if $F \neq 0$ then $R \cap F \neq 0 = dR \cap (R \cap F)$, contradicting the fact that D is a regular right Ore set in R. Hence, s_d is a unit in S. To see that $s_d^{-1} \in T$, let $i \in I$, and set $e = is_d^{-1}(1)$. If $e \neq 0$ then there exists $r \in R$ with $0 \neq er \in R$. Since D is a right Ore set in R, there exist $r_1 \in R$ and $d_1 \in D$ such that $rd_1 = dr_1$. Now, since $is_d^{-1}(dR) = 0$, we have $(er)d_1 = edr_1 = is_d^{-1}(dr_1) = 0$, contradicting the regularity of d_1 in R. Thus, $e = 0$, and $s_d^{-1} \in T$. It follows that $g(d)$ is a unit in T/I.

Using the right Ore condition on D and using the right common multiple property of D, it is now easy to see that the required ring Q can be taken to be the subring of T/I generated by $g(R) \cup \{g(d)^{-1} : d \in D\}$.

Uniqueness: Let (Q,f) be a right quotient ring of R relative to D. It is easy to see that (Q,f) has the following *universal property*: given any ring homomorphism $h : R \to Q'$ such that $h(d)$ is a unit in Q' for all $d \in D$, there exists a unique ring homomorphism $j : Q \to Q'$ such that $h = j \circ f$. The required map j is given by

$$j([f(r)][f(d)]^{-1}) = [h(r)][h(d)]^{-1} .$$

The uniqueness of (Q,f) follows trivially from its universal property. This completes the proof of the one-sided case.

Two-sided case: it suffices to observe that if D is a reversible Ore set in R then a right quotient ring of R relative to D is, in fact, a two-sided quotient ring of R relative to D. ◻

In view of the uniqueness assertion in Ore's theorem, it is customary to refer to any (right) quotient ring of R relative to D as *the* right quotient ring of R relative to D, and to denote it as R_D. Usually, R_D is viewed as an over-ring of $R/\rho_D(R)$ where $\rho_D(R) = \{r \in R \mid rd = 0 \text{ for some } d \in D\}$.

There is little that one can do about the occurrence of the right Ore condition in (1.1.1). However, as shown by (1.1.2), the right reversibility for right Ore sets comes for free in right Noetherian rings.

We need some terminology first.

Let M be a right module over a ring R, and let X and Y be non-empty subsets of R and M, respectively. The set

$$\{m \in M \mid mx = 0 \text{ for all } x \in X\}$$

is called the *left annihilator* of X in M, and is denoted as $l_M(X)$ or as $\mathrm{ann}_M X$. The *right annihilator* of Y in R is defined analogously, and is denoted as $r_R(Y)$.

When $M = R_R$, $l_R(X)$ is simply called the *left annihilator* of X, and $r_R(Y)$ is simply called the *right annihilator* of Y. Moreover, when R is clear from the context, $l_R(X)$ and $r_R(X)$ are often denoted as $l(X)$ and $r(Y)$, respectively.

A right ideal of R is called an *annihilator right ideal* if it is the right annihilator of a non-empty subset of R.

(1.1.2) *LEMMA*. In a ring R satisfying the ascending chain condition on annihilator right ideals, any right Ore set is right reversible, and any left regular right Ore set is regular.

Proof. Let D be a right Ore set in R. Suppose $dr = 0$ for some $d \in D$ and $r \in R$. The ascending chain condition on right annihilators yields $r(d^n) = r(d^{n+1})$ for some n, and the right Ore condition on D yields $d_1 \in D$ and $r_1 \in R$ such that $rd_1 = d^n r_1$. Thus, $0 = drd_1 = d^{n+1}r_1$, which yields $0 = d^n r_1 = rd_1$. Hence, D is right reversible in R.

The remaining assertion is now obvious. ☐

Now we have the following simpler form of Ore's Theorem for (right) Noetherian rings.

(1.1.3) *THEOREM*. Let D be a multiplicative set in a (right) Noetherian ring R. Then the (right) quotient ring of R relative to D exists iff D is a (right) Ore set in R.

Proof. Follows from (1.1.1) and (1.1.2). ☐

Later in the monograph, we often use (1.1.1) and (1.1.3) without explicit mention.

We remark that Ore's Theorem provides a helpful clarification rather than a workable solution of the problem of verifying the existence of a right quotient ring. For an arbitrary multiplicative set D in an arbitrary ring R, the Ore condition on D is no easier to verify than the existence of R_D. However, the rings of interest to us are at least right Noetherian (although they may be badly non-commutative), and the multiplicative sets of interest to us are usually 'natural' in some vague sense. Our hope of verifying the Ore condition for such sets is usually pinned on their presumed 'naturality'.

We provide an illustration of how 'naturality' of a multiplicative set can be helpful in verifying the Ore condition.

We recall that a *domain* is a (not necessarily commutative) ring with no non-zero zero-divisors. The non-zero elements of a domain R form a multiplicative set, which we denote as R^*.

A domain R is called a *(right) Ore domain* if R^* is a (right) Ore set in R.

It is obvious from Ore's Theorem that a domain R is a (right) Ore domain iff the (right) quotient ring R_{R^*} exists; in such a case, R_{R^*} is obviously a skew field, called the *(right) quotient skew field* of R.

A commutative domain is, trivially, an Ore domain. On the other hand, the free algebra on at least two generators over a field cannot be an Ore domain from either side.

The promised illustration is supplied by the following important class of Ore domains.

(1.1.4) *PROPOSITION*. Any (right) Noetherian domain is a (right) Ore domain.

Proof. Let R be a right Noetherian domain. If K and L are non-zero right ideals of R with $K \cap L = 0$ then it is easy to verify that, for any non-zero $d \in L$, $\sum_{n=0}^{\infty} d^n K$ is an infinite direct sum of non-zero right ideals of R; this contradicts the ascending chain condition on the right ideals of R. It follows that R* is a right Ore set in R. ☐

As in the Hilbert basis theorem, the polynomial ring in a finite number of central variables over a skew field is a Noetherian domain; cf., (1.3.14). Other natural examples of Noetherian domains are provided by the universal enveloping algebras of Lie algebras; cf., Dixmier (77;p.76), and by the group algebras of torsionfree polycyclic-by-finite groups; cf. Cliff (80), Farkas & Snider (76), and Passman (77).

A far-reaching generalization of (1.1.4), due to Goldie, forms the core of chapter 2; cf., (2.3.7).

QUOTIENT MODULES

We show that Ore's method can be extended to modules.

Let D be a multiplicative set in a ring R, and let M be a right R-module. Assume that a right quotient ring (Q,f) of R relative to D exists. Then a *right quotient module of M relative to D* is an ordered pair (K,g), where K is a right Q-module and $g : M \to K$ is an R-module homomorphism satisfying the following two conditions:

(1) For every $k \in K$, there exist $m \in M$ and $d \in D$ such that $k = [g(m)][f(d)]^{-1}$;

(2) $\ker g = \{m \in M \mid md = 0 \text{ for some } d \in D\}$.

In this context, it is customary to suppress the mention of g and to refer to K as a right quotient module of M relative to D or as an *Ore localization of M at D*.

As with the quotient rings, the preceding definition just states the desired outcome of the 'localization' of M at D. Fortunately, as shown by (1.1.5), the existence and the uniqueness of quotient modules present no new problems.

(1.1.5) *PROPOSITION*. Assume that a ring R has a right quotient ring relative to a multiplicative set D. Then, for any right R-module M, a right quotient module of M relative to D exists and is uniquely determined by M up to a natural isomorphism.

Proof. We recall from the proof of (1.1.1) that D is a right reversible right Ore set in R, that the set

$$\rho_D(R) = \{r \in R \mid rd = 0 \text{ for some } d \in D\}$$

is a two-sided ideal of R, and that the image $\bar{D}$ of D in $\bar{R} = R/\rho_D(R)$ is a regular right Ore set in $\bar{R}$. Now, set

$$\rho_D(M) = \{m \in M \mid md = 0 \text{ for some } d \in D\}.$$

A straightforward verification shows that $\rho_D(M)$ is an R-submodule of M, that $M\rho_D(R) \subseteq \rho_D(M)$, and that the elements of $\bar{D}$ are non-zero-divisors on the right $\bar{R}$-module $\bar{M} = M/\rho_D(M)$. A right quotient module M_D of M relative to D can now be constructed along the usual lines; cf., Jacobson (64).

Another method of constructing the needed module M_D goes as follows: let F be the $\bar{R}$-injective hull of $\bar{M}$; define M_D as

$$M_D = \{x \in F \mid xd \in \bar{M} \text{ for some } d \in \bar{D}\}.$$

It is not hard to verify that this works. Yet another method is to show that $M_D = M \otimes_R R_D$ works. We omit details.

The uniqueness part is trivial. ☐

In view of the uniqueness assertion in (1.1.5), it is customary to refer to any quotient module of M relative to D as *the* quotient module of M relative to D and denote it as M_D. Often, $M/\rho_D(M)$ is regarded as embedded in M_D.

RIGHT TORSION CLASSES

It is often convenient to think of localization in the broader context of torsion classes.

In this subsection, we define right torsion classes, and use them to obtain a convenient characterization of the right Ore condition.

A *right torsion class* ρ for a ring R is a non-empty class of right R-modules satisfying the following two conditions:

(1) The direct sum of any family of modules in ρ is also in ρ;

(2) For any exact sequence $0 \to M' \to M \to M'' \to 0$ of right R-modules, M belongs to ρ iff M' and M'' both belong to ρ.

Over a commutative domain, the modules which are torsion in the usual sense form a right torsion class. However, a right torsion class need not be tied to any such preconceived notion of 'torsion'.

We adapt the notions that are usually associated with 'torsion'.

Let ρ be a right torsion class for a ring R. For any right R-module M, the unique largest submodule of M belonging to ρ is called the *ρ-torsion submodule* of M and is denoted as $\rho(M)$. Naturally, M is called a *ρ-torsion module* if $\rho(M) = M$, and a *ρ-torsionfree module* if $\rho(M) = 0$.

Let N be a submodule of M. The full inverse image in M of $\rho(M/N)$ is called the *ρ-closure* of N in M and is denoted as $\rho\text{-cl}_M N$. The submodule N is said to be *ρ-closed* in M if $\rho\text{-cl}_M N = N$, and *ρ-dense* in M if $\rho\text{-cl}_M N = M$.

A ρ-closed submodule of R_R is called a *ρ-closed right ideal* of R. A *ρ-dense right ideal* of R is defined analogously.

There are three standard methods of manufacturing right torsion classes; these methods are indicated below.

The first one uses injective modules as follows. Let $\mathcal{F}$ be a non-empty class of right R-modules. Define

$$\rho_{\mathcal{F}} = \{M \in \text{mod-}R \mid \text{Hom}(M, E(F)) = 0 \text{ for all } F \in \mathcal{F}\}.$$

It is evident that $\rho_{\mathcal{F}}$ is a right torsion class for R. It is called the right torsion class *cogenerated* by $\mathcal{F}$. When $\mathcal{F}$ is a singleton set, say $\mathcal{F} = \{E\}$, the right torsion class $\rho_{\mathcal{F}}$ is denoted as ρ_E. It is not hard to see that any right torsion class ρ for R can be expressed as ρ_E for some injective module E. For instance,

$$E = \pi\{E(R/K) \mid K \text{ a } \rho\text{-closed right ideal of } R\}$$

will do.

The second method stems from the observation that the intersection of any collection of right torsion classes for R is also a right