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Diffeomorphisms and flows

1.1 Introduction

A dynamical system is one whose state changes with time (t). Two main types of dynamical system are encountered in applications: those for which the time variable is discrete ($t \in \mathbb{Z}$ or $\mathbb{N}$) and those for which it is continuous ($t \in \mathbb{R}$).

Discrete dynamical systems can be presented as the iteration of a function, i.e.

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t), \qquad t \in \mathbb{Z} \text{ or } \mathbb{N}. \tag{1.1.1}$$

When t is continuous, the dynamics are usually described by a differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}). \tag{1.1.2}$$

In (1.1.1 and 2), $\mathbf{x}$ represents the state of the system and takes values in the *state* or *phase space*. Sometimes the phase space is Euclidean space or a subset thereof, but it can also be a non-Euclidean structure such as a circle, a sphere, a torus or some other *differentiable manifold*.

In this chapter we will consider two special cases of the above equations, namely when:

(i) $\mathbf{f}$ in (1.1.1) is a *diffeomorphism*; and
(ii) the solutions of (1.1.2) can be described by a *flow* with velocity given by the *vector field* $\mathbf{X}$.

These two cases have been widely studied and they are fundamental to our understanding of dynamical systems. Smale, in his definitive work (Smale, 1967), pointed out that (i) and (ii) are closely related and our discussion emphasises this connection.

Any description of the theory of (i) and (ii) involves differentiable maps so let us begin by recalling some definitions. Let U be an open subset of $\mathbb{R}^n$. Then a function $g: U \to \mathbb{R}$ is said to be of *class* C^r if it is r-fold continuously differentiable, $1 \leqslant r \leqslant \infty$. Let V be an open subset of $\mathbb{R}^m$ and $\mathbf{G}: U \to V$. Given coordinates

$(x_1, \ldots, x_n)$ in U and $(y_1, \ldots, y_m)$ in V, $\mathbf{G}$ may be expressed in terms of component functions g_i: $U \to \mathbb{R}$, where

$$y_i = g_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, m. \tag{1.1.3}$$

The map $\mathbf{G}$ is called a *C^r-map* if g_i is C^r for each $i = 1, \ldots, m$. $\mathbf{G}$ is said to be *differentiable* if it is a C^r-map for some $1 \leqslant r \leqslant \infty$ and to be *smooth* if it is C^∞. Maps that are continuous but not differentiable are, conventionally, referred to as *C^0-maps.*

Definition 1.1.1 $\mathbf{G}$ *is said to be a* diffeomorphism *if it is a bijection and both* $\mathbf{G}$ *and* $\mathbf{G}^{-1}$ *are differentiable mappings.* $\mathbf{G}$ *is called a* C^k-diffeomorphism *if both* $\mathbf{G}$ *and* $\mathbf{G}^{-1}$ *are* C^k*-maps.*

Observe that the bijection $\mathbf{G}$: $U \to V$ is a diffeomorphism if and only if $m = n$ and the matrix of partial derivatives

$$\mathbf{DG}(x_1, \ldots, x_n) = \left[\frac{\partial g_i}{\partial x_j}\right]_{i,j=1}^{n} \tag{1.1.4}$$

is non-singular at every $\mathbf{x} \in U$. Thus $\mathbf{G}(x, y) = (\exp(y), \exp(x))^{\mathrm{T}}$ with $U = \mathbb{R}^2$ and $V = \{(x, y) | x, y > 0\}$ is a diffeomorphism because Det $\mathbf{DG}(x, y) = -\exp(x + y) \neq 0$ for each $(x, y) \in \mathbb{R}^2$.

If $\mathbf{G}$ satisfies Definition 1.1.1 with $\mathbf{G}$ and $\mathbf{G}^{-1}$ continuous, rather than differentiable, maps then $\mathbf{G}$ is said to be a *homeomorphism*. As we shall see, such maps play a central role in the topological theory of flows and diffeomorphisms.

The above definitions are adequate provided phase space is Euclidean, but, as we have already mentioned, the natural setting for dynamics is a *differentiable manifold*. The important point here is that manifolds have the property that they are 'locally Euclidean' and this allows us to extend the idea of differentiability to functions defined on them. If M is a manifold of dimension n then, for any $\mathbf{x} \in M$, there is a neighbourhood $W \subseteq M$ containing $\mathbf{x}$ and a homeomorphism $\mathbf{h}$: $W \to \mathbb{R}^n$ which maps W onto a neighbourhood of $\mathbf{h}(\mathbf{x}) \in \mathbb{R}^n$. Since we can define coordinates in $U = \mathbf{h}(W) \subseteq \mathbb{R}^n$ (the coordinate curves of which can be mapped back onto W), we can think of $\mathbf{h}$ as defining local coordinates on the patch W of M (see Figure 1.1).

The pair $(U, \mathbf{h})$ is called a *chart* and we can use it to give meaning to differentiability on W. Let us assume, for simplicity, that $\mathbf{f}$: $W \to W$, then $\mathbf{f}$ induces a map $\tilde{\mathbf{f}} = \mathbf{h} \cdot \mathbf{f} \cdot \mathbf{h}^{-1}$: $U \to U$ (see Figure 1.2). We say that $\mathbf{f}$ is a C^k-map on W if $\tilde{\mathbf{f}}$ is a C^k-map on U. This construction allows us to give a definition of a *local* diffeomorphism on M.

In order to obtain a global description of the manifold, we cover it with a family of open sets, W_α, each with its associated chart $(U_\alpha, \mathbf{h}_\alpha)$ (predictably, the set of all charts is called an *atlas*). If $W_\alpha \cap W_\beta$ is not empty, then either $(U_\alpha, \mathbf{h}_\alpha)$ or $(U_\beta, \mathbf{h}_\beta)$ can be used to provide local coordinates for $W_\alpha \cap W_\beta$. This possibility induces overlap maps, $\mathbf{h}_{\alpha\beta}$ and $\mathbf{h}_{\beta\alpha}$ between $\mathbf{h}_\alpha(W_\alpha \cap W_\beta) \subseteq U_\alpha$ and $\mathbf{h}_\beta(W_\alpha \cap W_\beta) \subseteq U_\beta$ (see

Figure 1.1 Examples of differentiable manifolds and some 'patches' of local coordinates. Several open sets based on patches of this kind may be required in order to cover the whole manifold.

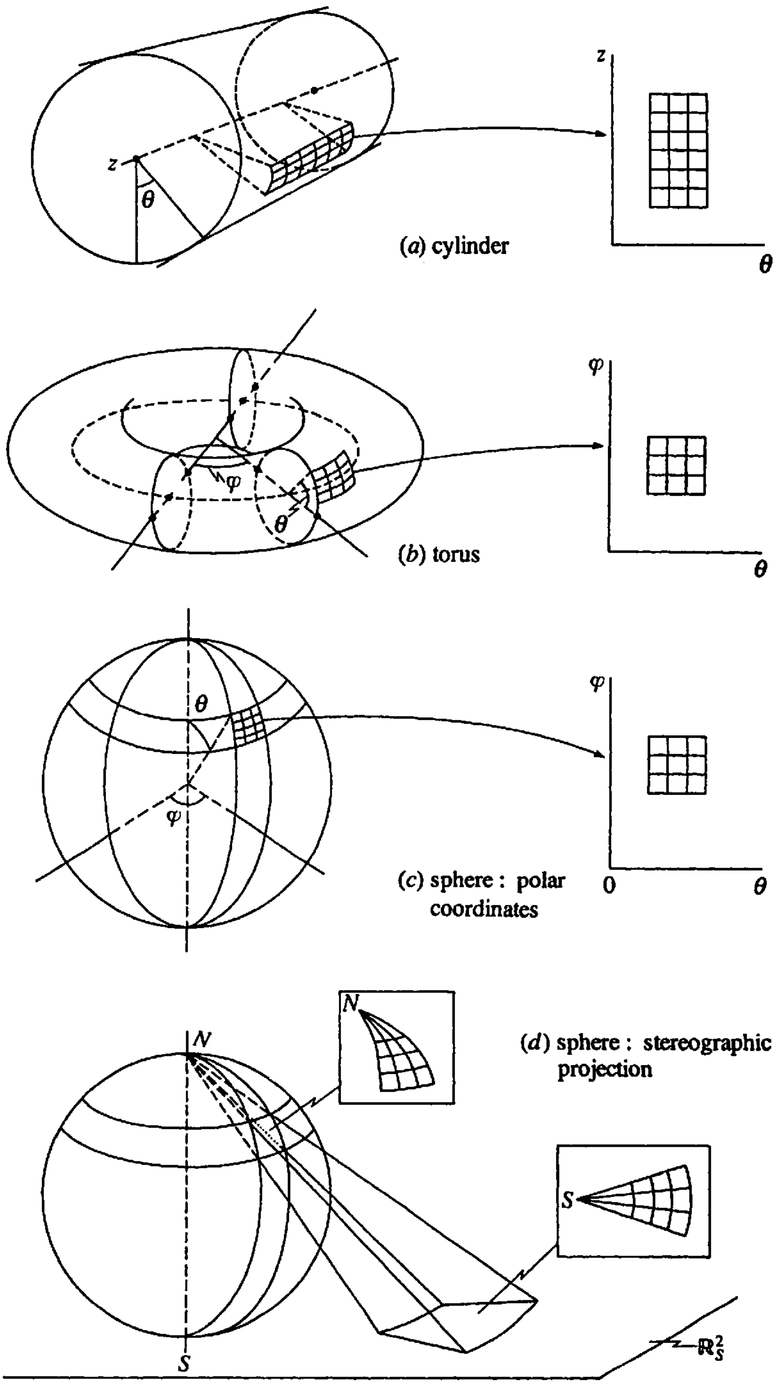


Figure 1.3). If we now consider $\mathbf{f}\colon W_\alpha \cap W_\beta \to W_\alpha \cap W_\beta$, we have two alternative representatives $\tilde{\mathbf{f}}_\alpha = \mathbf{h}_\alpha \cdot \mathbf{f} \cdot \mathbf{h}_\alpha^{-1}$ and $\tilde{\mathbf{f}}_\beta = \mathbf{h}_\beta \cdot \mathbf{f} \cdot \mathbf{h}_\beta^{-1}$ for $\mathbf{f}$. Since $\tilde{\mathbf{f}}_\alpha$ and $\tilde{\mathbf{f}}_\beta$ are determined by different charts, they might belong to different differentiability classes, so that the class of $\mathbf{f}$ would be ambiguous. A manifold is said to be *differentiable* if all the overlap maps are diffeomorphisms of the same differentiability class, C^r say. Now, from Figure 1.3,

$$\begin{aligned}\tilde{\mathbf{f}}_\beta &= \mathbf{h}_\beta \cdot \mathbf{f} \cdot \mathbf{h}_\beta^{-1} \\ &= (\mathbf{h}_\beta \cdot \mathbf{h}_\alpha^{-1}) \cdot (\mathbf{h}_\alpha \cdot \mathbf{f} \cdot \mathbf{h}_\alpha^{-1}) \cdot (\mathbf{h}_\alpha \cdot \mathbf{h}_\beta^{-1}) \\ &= \mathbf{h}_{\alpha\beta} \cdot \tilde{\mathbf{f}}_\alpha \cdot \mathbf{h}_{\alpha\beta}^{-1}. \end{aligned} \tag{1.1.5}$$

Thus all local representatives of $\mathbf{f}$ have the same differentiability class, C^k say, with $k \leqslant r$. It is important to note that r is determined entirely by the charts and hence by the structure of M. A manifold with overlap maps of class C^r is called a *C^r-manifold.*

The discussion presented above is, of course, incomplete. We have only considered maps taking a chart into itself. This is clearly not true in general. Given $\mathbf{f}\colon M \to M$, then $\mathbf{f}\colon W_\alpha \to W_\beta$ and $\mathbf{f}\colon W_\alpha \cap W_\gamma \to W_\beta \cap W_\delta$. The generalisation of our simple arguments that allows for these omissions is considered in Exercise 1.1.2. Needless to say, the 'message' is unchanged by these manipulations.

A more detailed discussion of differentiable manifolds is not necessary here (the interested reader should consult Arnold (1973) or Chillingworth (1976)). While the ideas outlined above provide valuable background knowledge, we will rarely find ourselves involved with charts, atlases, etc. This is because our concern is the *dynamics* of maps defined on M *given* that they are diffeomorphisms or flows.

Figure 1.2 Commutative diagram illustrating the representation of $\mathbf{f}$ defined on an open set W of M in a local shart $(U, \mathbf{h})$.

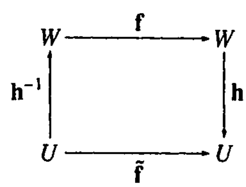
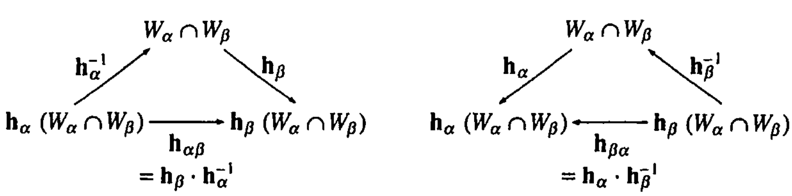


Figure 1.3 Illustration of the definition of the overlap maps $\mathbf{h}_{\alpha\beta}$ and $\mathbf{h}_{\beta\alpha}$. Note that $\mathbf{h}_{\beta\alpha} = \mathbf{h}_{\alpha\beta}^{-1}$.



These maps are usually presented to us in local coordinates so that the manifold structure does not appear explicitly.

1.2 Elementary dynamics of diffeomorphisms

1.2.1 Definitions

Let M be a differentiable manifold and suppose $\mathbf{f}\colon M \to M$ is a diffeomorphism. For each $\mathbf{x} \in M$, the iteration (1.1.1) generates a sequence, the distinct points of which define the *orbit* or *trajectory* of $\mathbf{x}$ under $\mathbf{f}$. More precisely, the orbit of $\mathbf{x}$ under $\mathbf{f}$ is $\{\mathbf{f}^m(\mathbf{x}) | m \in \mathbb{Z}\}$. For $m \in \mathbb{Z}^+$, $\mathbf{f}^m$ is the composition of $\mathbf{f}$ with itself m times. Since $\mathbf{f}$ is a diffeomorphism $\mathbf{f}^{-1}$ exists and $\mathbf{f}^{-m} = (\mathbf{f}^{-1})^m$. Finally, $\mathbf{f}^0 = \mathbf{id}_M$, the identity map on M. Typically, the orbit of $\mathbf{x}$ is a bi-infinite sequence of distinct points of M. However, there are two important exceptions to this state of affairs.

Definition 1.2.1 *A point* $\mathbf{x}^* \in M$ *is called a* fixed point *of* $\mathbf{f}$ *if* $\mathbf{f}^m(\mathbf{x}^*) = \mathbf{x}^*$ *for all* $m \in \mathbb{Z}$.

Definition 1.2.2 *A point* $\mathbf{x}^* \in M$ *is a* periodic point *of* $\mathbf{f}$ *if* $\mathbf{f}^q(\mathbf{x}^*) = \mathbf{x}^*$, *for some integer* $q \geqslant 1$.

The least value of q satisfying Definition 1.2.2 is called the *period* of the point $\mathbf{x}^*$ and the orbit of $\mathbf{x}^*$, i.e.

$$\{\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*), \ldots, \mathbf{f}^{q-1}(\mathbf{x}^*)\}, \tag{1.2.1}$$

is said to be a *periodic orbit of period* q or a q*-cycle* of $\mathbf{f}$. Clearly, since $\mathbf{f}^q(\mathbf{x}^*) = \mathbf{x}^*$, it is the sequence $\{\mathbf{f}^m(\mathbf{x}^*)\}_{m=-\infty}^{\infty}$ which is q-periodic. Notice that a fixed point is a periodic point of period one and a periodic point of $\mathbf{f}$ with period q is a fixed point of $\mathbf{f}^q$. Morever, if $\mathbf{x}^*$ is a periodic point of period q for $\mathbf{f}$ then so are all of the other points in the orbit of $\mathbf{x}^*$. For example, if $\mathbf{f}^q(\mathbf{x}^*) = \mathbf{x}^*$ then $\mathbf{f}(\mathbf{f}^q(\mathbf{x}^*)) = \mathbf{f}(\mathbf{x}^*) = \mathbf{f}^q(\mathbf{f}(\mathbf{x}^*))$ and $\mathbf{f}(\mathbf{x}^*)$ is therefore a periodic point of period q, and so on for $\mathbf{f}^2(\mathbf{x}^*), \ldots, \mathbf{f}^{q-1}(\mathbf{x}^*)$.

Fixed and periodic points can be classified according to the behaviour of the orbits of points in their vicinity. The following ideas are due to Liapunov.

Definition 1.2.3 *A fixed point,* $\mathbf{x}^*$, *is said to be* stable *if, for every neighbourhood* N *of* $\mathbf{x}^*$, *there is a neighbourhood* $N' \subseteq N$ *of* $\mathbf{x}^*$ *such that if* $\mathbf{x} \in N'$ *then* $\mathbf{f}^m(\mathbf{x}) \in N$ *for all* $m > 0$.

Essentially, Definition 1.2.3 says that iterates of points 'near to' a stable fixed point, remain 'near to' it for $m \in \mathbb{Z}^+$. If a fixed point $\mathbf{x}^*$ is stable and $\lim_{m \to \infty} \mathbf{f}^m(\mathbf{x}) = \mathbf{x}^*$, for all $\mathbf{x}$ in some neighbourhood of $\mathbf{x}^*$, then the fixed point is said to be *asymptotically stable*. Trajectories of points near to an asymptotically stable fixed point move toward it as m increases. Fixed points that are stable, but not asymptotically stable, are said to be *neutrally* or *marginally* stable and those that are not stable in the sense of Definition 1.2.3 are *unstable.*

1.2.2 Diffeomorphisms of the circle

The circle (S^1) is arguably the simplest non-Euclidean differentiable manifold. It is compact (see Chillingworth, 1976, p. 143) so 'behaviour at infinity' is not a problem; it has no boundary so that dynamics can be studied without the complication of boundary conditions on the functions concerned and it is one-dimensional. The dynamics of diffeomorphisms on the circle therefore provide an ideal opportunity for us to illustrate the definitions given in § 1.2.1.

Some of the simplest examples of diffeomorphisms on S^1 are the pure rotations. They are easily defined in terms of the angular displacement (θ) at the centre of the circle relative to a reference radius (see Figure 1.4). In terms of this local coordinate, an anticlockwise rotation by α may be written as

$$R_\alpha(\theta) = (\theta + \alpha) \bmod 1. \tag{1.2.2}$$

Here we have assumed that θ is measured in units of 2π. If $\alpha = p/q$, $p, q \in \mathbb{Z}$ and relatively prime, then

$$R_\alpha^q(\theta) = (\theta + p) \bmod 1 = \theta \tag{1.2.3}$$

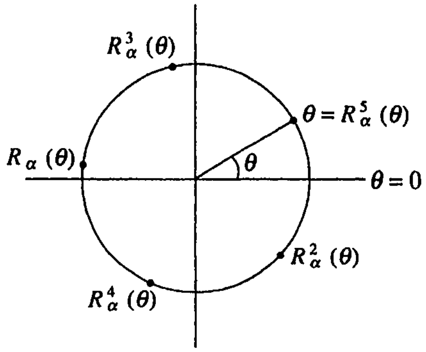
and we conclude (cf. Definition 1.2.2) that every point of the circle is a periodic point of period-q, i.e. the orbit of any point is a q-cycle (see Figure 1.4). If α is irrational then

$$R_\alpha^m(\theta) = (\theta + m\alpha) \bmod 1 \neq \theta, \tag{1.2.4}$$

for any θ and, in fact, the orbit of any point fills the circle densely (see Exercise 1.2.1).

Obviously more general diffeomorphisms of S^1 do not simply rotate all points uniformly. Crudely speaking they compress some arcs of the circle and stretch

Figure 1.4 Typical orbit of the pure rotation R_α for $\alpha = p/q = 2/5$. Observe that the orbit of θ winds around the circle $p = 2$ times before returning to θ on the fifth iteration.



others. It is then difficult to recognise fixed or periodic points from the representation of orbits on the circle itself. This is a problem for any map (f) of the circle, whether it is a diffeomorphism or not, and it is solved by considering a *lift* of f.

The natural setting for introducing the lift of $f: S^1 \to S^1$ is when f is a homeomorphism rather than a diffeomorphism and it would be perverse to artificially confine our discussion to the differentiable case. Moreover, by taking f to be a homeomorphism at this point we can better appreciate the consequences of imposing differentiability on f and f^{-1}. Thus, let $f: S^1 \to S^1$ be a homeomorphism and suppose there is a continuous function $\bar{f}: \mathbb{R} \to \mathbb{R}$ such that

$$\pi(\bar{f}(x)) = f(\pi(x)) \tag{1.2.5}$$

(see Figure 1.5), where

$$\pi(x) = x \bmod 1 = \theta. \tag{1.2.6}$$

Then $\bar{f}$ is called a *lift of* $f: S^1 \to S^1$ onto $\mathbb{R}$.

Proposition 1.2.1 *Let $\bar{f}$ be a lift of the orientation-preserving homeomorphism $f: S^1 \to S^1$. Then*

$$\bar{f}(x+1) = \bar{f}(x) + 1 \tag{1.2.7}$$

for every $x \in \mathbb{R}$.

Proof. Observe that

$$f(\pi(x)) = f(\pi(x+1)) \tag{1.2.8}$$

because $\pi(x) = \pi(x+1)$ by (1.2.6). If we substitute for $f \cdot \pi$ from (1.2.5), (1.2.8) becomes

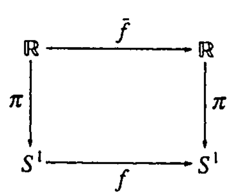
$$\pi(\bar{f}(x)) = \pi(\bar{f}(x+1)) \tag{1.2.9}$$

and it follows that

$$\bar{f}(x+1) = \bar{f}(x) + k(x), \tag{1.2.10}$$

where $k(x)$ is an integer possibly depending on x. However, since $\bar{f}$ is continuous, $k(x)$ must be continuous and this is only possible if $k(x) \equiv k \in \mathbb{Z}$.

Figure 1.5 Commutative diagram illustrating the definition of the lift of a circle homeomorphism f. The map π takes infinitely many equivalent points of $\mathbb{R}$ onto a single point of S^1.



Suppose $k \geqslant 2$, then $\bar{f}(x)$ and $\bar{f}(x+1)$ differ by more than two and $\bar{f}$ takes the form shown schematically in Figure 1.6(*a*). Clearly, the points x_0 and x_1 satisfying $\bar{f}(x_0) = 1$ and $\bar{f}(x_1) = 2$ are both less than unity. This means that π maps them to distinct points on S^1. However, $\bar{f}(x_0)$ and $\bar{f}(x_1)$ differ by unity and therefore represent the same point on S^1. This contradicts the hypothesis that f is a homeomorphism. Hence $k \leqslant 1$.

If $k = 0$, $\bar{f}(0) = \bar{f}(1)$ and $\bar{f}$ fails to be injective on (0, 1) (see Figure 1.6(*b*)). Again this contradicts the fact that f is a homeomorphism.

If $k < 0$ then continuity of $\bar{f}$ can only be maintained if f is orientation-reversing in contradiction to hypothesis. Moreover, it is clear that similar arguments would lead to a minus sign in the right hand side of (1.2.7) for orientation-reversing f.

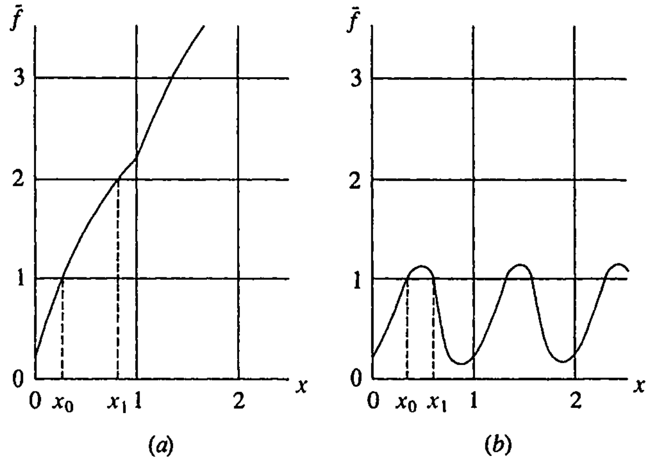
Finally, we conclude that $k = 1$ and (1.2.7) follows. □

It is important to realise that not every continuous function satisfying (1.2.7) is the lift of some homeomorphism. The function shown in Figure 1.7 is continuous and satisfies (1.2.7) but fails to be the lift of a homeomorphism because it is not injective. Figure 1.7 also highlights the geometrical significance of (1.2.7); namely that the graph of $\bar{f}$ in the interval $[k, k+1]$ is obtained by shifting the graph of $\bar{f}$ in [0, 1] vertically by k units. In this way any continuous function g, defined on [0, 1], that is injective, and such that $g(1) = g(0) + 1$, can be used to construct a lift $\bar{f}$ for some homeomorphism $f: S^1 \to S^1$. The function f is given by (1.2.5). A simple example of this construction is given in Figure 1.8(*a*) where

$$g(x) = -x^2 + 2x + \tfrac{1}{2}, \qquad (1.2.11)$$

$x \in [0, 1]$. In this case, $\bar{f}$ is a continuous bijection but it is not differentiable at $x = 1, 2, \ldots$. This reflects on the corresponding f which is a homeomorphism,

Figure 1.6 Schematic forms for $\bar{f}$ when (1.2.10) has (*a*) $k = 2$; (*b*) $k = 0$. In both cases, the hypothesis that f is a homeomorphism is contradicted.



but not a diffeomorphism of S^1. To obtain the latter, $\bar{f}$ must be a bijection and differentiable for all $x \in \mathbb{R}$. An example of this type is shown in Figure 1.8(*b*) where

$$g(x) = x + \tfrac{1}{2} + \tfrac{1}{10} \sin 2\pi x, \qquad (1.2.12)$$

$x \in [0, 1]$.

Notice, we have, without loss of generality, taken $\bar{f}(0) \in [0, 1)$ in both of the above examples. Observe that, $\pi(\bar{f}(x) + k) = \pi(\bar{f}(x))$, for any $k \in \mathbb{Z}$. Thus if $\bar{f}(x)$ is a lift of f then so is $\bar{f}_k(x) = \bar{f}(x) + k$, $k \in \mathbb{Z}$. Therefore, unless otherwise stated, we will assume that $\bar{f}$ is the member of this family of lifts satisfying $\bar{f}(0) \in [0, 1)$.

Figure 1.7 The function $\bar{f}$ shown here cannot be the lift of a homeomorphism $f: S^1 \to S^1$ because it is not injective.

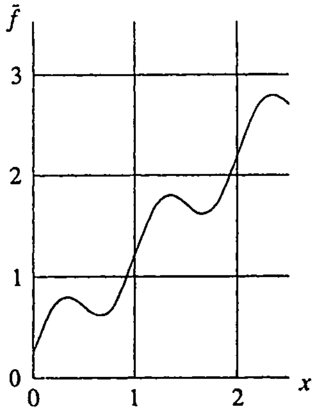
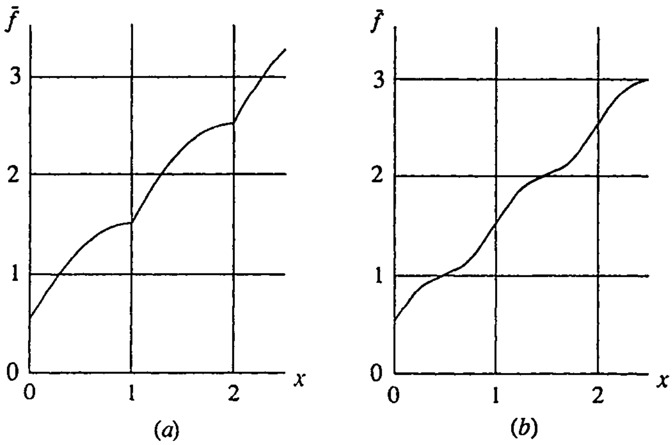


Figure 1.8 The function shown in (*a*) is the lift of a homeomorphism, but not of a diffeomorphism, of the circle. Lifts of diffeomorphisms are differentiable functions of x, see (*b*) for example, where $\bar{f}$ is obtained from (1.2.12).



How are the fixed or periodic points of $f: S^1 \to S^1$ related to the properties of the lift $\bar{f}$?

Proposition 1.2.2 *Let $f: S^1 \to S^1$ be an orientation-preserving homeomorphism and suppose that $\bar{f}$ is the lift of f with $\bar{f}(0) \in [0, 1)$. Then $\pi(x^*)$ is a fixed point of f if and only if either*

$$\bar{f}(x^*) = x^* \tag{1.2.13a}$$

or

$$\bar{f}(x^*) = x^* + 1. \tag{1.2.13b}$$

Proof. If $\bar{f}(x^*) = x^*$ (or $\bar{f}(x^*) = x^* + 1$) then

$$\pi(\bar{f}(x^*)) = \pi(x^*) \qquad (\text{or} \quad \pi(\bar{f}(x^*)) = \pi(x^* + 1) = \pi(x^*)). \tag{1.2.14}$$

In either case,

$$f(\pi(x^*)) = \pi(x^*) \tag{1.2.15}$$

by (1.2.5) and $\pi(x^*)$ is a fixed point of f.

If $\theta^* = \pi(x^*)$ is a fixed point of f, i.e. $f(\theta^*) = \theta^*$, then

$$f(\pi(x^*)) = \pi(x^*) = \pi(\bar{f}(x^*)) \tag{1.2.16}$$

by (1.2.5). Thus

$$\bar{f}(x^*) = x^* + k, \qquad k \in \mathbb{Z}. \tag{1.2.17}$$

Let $x^* = y^* + l$, $l \in \mathbb{Z}$, $y^* \in [0, 1)$ then (1.2.17) becomes

$$\bar{f}(y^*) + l = y^* + l + k. \tag{1.2.18}$$

Here we have noted that a simple induction on $\bar{f}(x+1) = \bar{f}(x) + 1$ gives $\bar{f}(x+l) = \bar{f}(x) + l$. Thus, if (1.2.17) is satisfied for any x^*, it must be satisfied for a point $y^* \in [0, 1)$. Now, $\bar{f}(1) = \bar{f}(0) + 1$ and $\bar{f}$ is injective so that $\bar{f}(0) \leqslant \bar{f}(y) < \bar{f}(0) + 1$ for $y \in [0, 1)$. Therefore, (1.2.18) cannot be satisfied unless $k = 0$ or 1 (see Figure 1.9). □

Proposition (1.2.2) can be used to locate periodic points of f. Suppose that f has lift $\bar{f}$, i.e. $\pi(\bar{f}(x)) = f(\pi(x))$, $x \in \mathbb{R}$, then

$$\pi(\bar{f}^2(x)) = \pi(\bar{f}(\bar{f}(x))) = f(\pi(\bar{f}(x)) = f^2(\pi(x)). \tag{1.2.19}$$

Thus $\bar{f}^2$ is a lift of f^2. It only remains to ensure that $\bar{f}^2(0) \in [0, 1)$ (i.e. choose the lift $\bar{f}^2 - [\bar{f}^2(0)]$, where $[\cdot]$ denotes the integer part of $\cdot$), and Proposition 1.2.2 allows us to find the period-2 points of f. These arguments obviously extend to points of period $q > 2$.

An alternative approach is to recognise that if $\bar{f}^q(0) \in [l, l+1)$ then (1.2.13) is