INTRODUCTION

It has long been conjectured that if the finite group G acts freely on the standard sphere S^3, then the action is topologically conjugate to a free linear action. Equivalently the orbit space S^3/G is homeomorphic to a manifold of constant positive curvature, and such elliptic 3-manifolds are classified in terms of the fixed point free representations of G in SO(4), see the book by J. Wolf [Wo] for example. The purpose of these notes is to collect together the evidence in favour of this conjecture at least for the class of groups G which are known to act freely and linearly in dimension three. The main result is that if G is solvable and acts freely on S^3 in such a way that the action restricted to all cyclic subgroups of odd order is conjugate to a linear action, then the action of G is conjugate to a linear action. This reduction to cyclic groups (which is false in higher dimensions) depends on (a) the algebraic classification of the fundamental groups of elliptic manifolds and (b) geometric arguments due to R. Myers and J. Rubinstein classifying free Z/2 and Z/3-actions on certain Seifert fibre spaces. The proof is contained in Chapters I-IV; for part (a) we follow an unpublished joint manuscript with C.T.C. Wall [Th-W], and for part (b) the original papers [My] and [R2]. Besides the reduction theorem already quoted the argument implies that the original conjecture holds for groups G whose order is divisible by the primes 2 and 3 only. For the non-solvable group $SL(2,F_5)$ the corresponding reduction theorem is weaker - a free action which is linear on each element embeds in a free linear action on S^7 (see Chapter VI).

The other topics which we consider are the classifying map $B\phi: BG \to BDiff^+S^3$ associated with a free smooth action by G, the homotopy classes of finite 3-dimensional Poincaré complexes with finite fundamental group, and (in an appendix) Heegard decompositions of genus 2 for elliptic manifolds. In a "concluding unscientific postscript" we suggest various ways in which the remaining core problem of free actions by cyclic groups may be approached - but the actual results we obtain are very weak.

These notes are based on a course of lectures which I gave at the University of Chicago in the spring of 1983, and have been available in a preliminary version for some time. Among those who listened to me then I am particularly grateful to Peter May and Dick Swan for their helpful comments. I would also like to thank Terry Wall for teaching me over the years much of the mathematics on which this work is based, and for being always willing to listen to my ideas however haltingly expressed. Finally I would like to thank the Editor of the LMS Lecture Notes for agreeing to accept an expanded version of the Chicago notes for publication in the series, David Tranah of Cambridge University Press for his advice and patience, and Gwen Jones for typing the manuscript.

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CHAPTER I: SEIFERT MANIFOLDS.

Let M^3 be a compact, connected 3-dimensional manifold without boundary. Where necessary we shall assume that M^3 has a smooth structure - there is no loss of generality in doing so, since M^3 is triangulable and the obstructions to smoothing vanish. Consider first a smooth action by the compact group $SO(2) = S^1$ on M^3. We use the notation

$$G \times M \to M, \quad x \longmapsto gx,$$

subject to the conditions (i) $g_1(g_2x) = (g_1g_2)x$, (ii) $1x = x$ and (iii) if $gx = x$ for all $x \in M$, then $g = 1$. Under condition (iii) the action of G is said to be effective. The orbit $Gx = \{gx: g \in G\}$ is homeomorphic to the homogeneous space G/G_x, where $G_x = \{g \in G: gx = x\}$ is the isotropy group of x. Since G is abelian, G_x is the isotropy group of each point of the orbit, and $\bigcap_{x \in M} G_x = \{1\}$ by condition (iii). The space of orbits $M^* = M/_G$ is a 2-dimensional manifold with respect to the quotient topology; the discussion of isotropy below will make this plain. Since G acts on the tangent space to x via the differential there is a representation of G_x on the normal space to x, which may be identified with the complement in T_x to the tangent space along the orbit. With respect to some equivariant Riemanian metric let V_x be the unit disc of this "slice" representation space. The equivariant classification theorem below for pairs (M^3, S^1) depends on two classical results from equivariant topology, see for example [J]:

THEOREM 1.1 The total space of the disc bundle $G \times_{G_x} V_x$ is equivariantly diffeomorphic to a G-invariant tubular neighbourhood of the orbit Gx in M, under the map $[g,v] \mapsto gv$, and the zero section $G/_{G_x}$ maps to the orbit Gx .

THEOREM 1.2 (stated for abelian transformation groups). Let G act smoothly and effectively on the connected manifold M. Then there is a subgroup $H \hookrightarrow G$ such that the union of the orbits with H as isotropy subgroup forms a dense subset of M. Furthermore the orbit space of these so called principal orbits is connected.

H is called the principal isotropy subgroup; the union $M_{(H)}$ of the principal orbits has the structure of a fibre bundle. The first theorem depends on the choice of an equivariant Riemannian metric on M, which gives rise to an exponential map of maximal rank near the zero section $G/_{G_x}$ of the normal bundle. Since the manifold M is compact, it is enough to prove Theorem 1.2 for the submanifold $G \times_{G_x} V_x$. Here the principal orbits belong to the complement of the zero section, a point is moved in the direction of Gx by G, and in the normal direction by G_x (modulo the kernel of the slice representation).

For the pair (M^3, S^1) we see that a closed subgroup is either $\{1\}$, S^1 or isomorphic to the finite cyclic group $Z/_\mu$. The principal orbit type equals $\{1\}$, M* is a 2-manifold, in general with boundary. However we shall restrict attention to the case when $M^* = \emptyset$, when M* is characterised by the pairs (o_1, g) or (n_1, g). The first symbol distinguishes between orientable and non-orientable; the second is the genus. The assumption that $M^* = \emptyset$ eliminates discussion of (a) fixed points (isotropy subgroup equals S^1) and (b) $G_x = Z/_2$ with the slice action equal to reflection about an arc. Theorem 1.1 shows that the exceptional orbits map to a finite union of r distinct points in M*.

Consider an exceptional orbit, $G_x = Z/_\mu$ with $\mu > 1$ and case (b) excluded. Identify a slice with the 2-disc D^2, and let $\zeta = 2\pi/_\mu$ act via

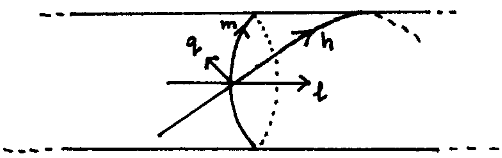
$$\zeta(r,\theta) = (r,\theta+\nu\zeta), \text{ where } (\nu,\mu) = 1 \;\&\; 0 < \nu < \mu.$$

The action inside a small tubular neighbourhood N of the orbit can now (following 1.1) be written as $(r,\theta,\psi) \quad (r,\theta + \nu\zeta, \phi + \mu\psi)$, where ψ denotes the coordinate on S^1. The exceptional orbit itself corresponds to $r = \theta = 0$ and has isotropy group of order μ. The action on N is completely determined by the

Seifert invariants (α,β), where $\alpha = \mu$, $\beta\nu \equiv 1 \pmod{\alpha}$ and $0 < \beta < \alpha$. Changing the orientation of the pair $(N,\partial N)$ while keeping the S^1-action fixed replaces the pair (α,β) by $(\alpha,\alpha-\beta)$. With computation of the fundamental group in mind pick out a curve q in ∂N, which is orthogonal to a principal orbit h on ∂N, and given by

$$q = \{(r,\theta,\phi): r = 1.\ \theta = \rho\chi,\ \phi = \beta\chi.\ 0 \le \chi < 2\pi\}.$$

Here orientation is according to decreasing χ, $\rho = (\beta\nu-1)/\alpha$, and we note that the curve $m = \alpha q + \beta h$ is null-homotopic in the solid torus N, see diagram.



Remark: if no particular orientation of N is specified, there is an ambiguity in the Seifert invariants, unless we take $0 < \beta \le \frac{\alpha}{2}$.

When there are no exceptional orbits, that is the S^1-action is principal, the bundle $M \to M^*$ is classified by an "obstruction or first Chern class" $b \in H^2(M^*,Z)$ Z or $Z/_2$, depending on whether M^* is orientable or non-orientable. Again with the computation of π_1 in mind one can interpret b geometrically as follows:- write $M_o = M - \overset{\circ}{N}_o$ and let q_o be a cross-section (as above) to the action on the boundary. Then with a suitable orientation convention the equivariant sewing of M_o to N_o along ∂N_o is determined up to equivariant diffeomorphism by making $q_o + bh$ into a meridian curve, that is one null-homotopic in N_o. In the general case one defines $b \in H^2(M^* - (D^2_1 \cup \dots \cup D^2_r)^\circ, S^1_1 \cup \dots \cup S^1_r, Z)$.

THEOREM 1.3 *Let* S^1 *act effectively and smoothly on a closed, connected, compact* C^∞*-manifold* M^3, *and assume that the orbit manifold* M^* *is without boundary. Then up to equivarient diffeomorphism* M^3 *is determined by the orbit invariants*

$$\{b;\ (\varepsilon, g);\ (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\},$$

subject to the conditions (i) $b \in Z$ *if* $\varepsilon = o_1$ & $b \in Z/_2$ *if* $\varepsilon = n_1$,

(ii) $0 < \beta_j < \alpha_j$, $(\alpha_j, \beta_j) = 1$ if $\varepsilon = o_1$, *and* $0 < \beta_j \leq \frac{\alpha_j}{2}$,

$(\alpha_j, \beta_j) = 1$ *if* $\varepsilon = n_1$.

Furthermore, if $\varepsilon = o_1$ *the equivariant diffeomorphism preserves orientation.*

Proof. It is not too hard given the discussion above to use a given family of invariants to build up an S^1-action. Conversely, given (M^3, S^1) classify M^* by (ε, g). Theorem 1.2

implies that we can find mutually disjoint tubular neighbourhoods of the finite family of exceptional orbits $E_1, \ldots, E_r$. As above each of these can be described by the pair (α_j, β_j), subject to orientation conventions. The class b is now the obstruction associated to the bundle of principal orbits. Given two manifolds with matching invariants it is also clear that we can step-by-step construct an equivariant diffeomorphism between them.

Remark: when $\varepsilon = o_1$, $-M$ is specified by $\{-b-r; (o_1, g); (\alpha_1, \alpha_1 - \beta_1) \ldots (\alpha_r, \alpha_r - \beta_r)\}$.

Example: $M^3 = (D_1^2 \times S_1^1) \cup_F (D_2^2 \times S_2^1)$, where (a) F is the diffeomorphism of the common boundary defined by the matrix $\begin{pmatrix} -1 & 0 \\ -m & 1 \end{pmatrix}$, (b) S^1 acts trivially on D_i^2 and by rotation on S_i^1. Then F is equivariant (exercise) and the action has no exceptional orbits. The manifold M^3 coincides with the lens space $L^3(-m, 1) = L^3(m, m-1)$ and the orbit invariants are $\{-m; (o_1, 0)\}$. The reader may find this clearer having read the section below on the fundamental group, when it will also emerge that the same space may correspond to more than one family of orbit invariants. Obviously two spaces may be diffeomorphic without being equivariantly diffeomorphic.

Motivated by the classification theorem 1.3 for pairs (M^3, S^1) we recall the original definition of a Seifert Manifold as a manifold M^3 which (a) decomposes into a collection of simple closed curves (called fibres), in such a

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way that each point belongs to one and only one fibre and (b) each fibre has a tubular neighbourhood N, consisting of fibres, such that $N = D^2 \times S^1/Z/\mu$ with the cyclic group Z/μ acting as in the discussion of exceptional orbits.

Such a manifold is the total space of a Seifert bundle – these are defined analogously to fibre bundles, except that the local product structure must be weakened to allow orbit spaces like N above. The coordinate transformations are given by maps γ_{ij} ; $\mathring{D}_i \cap \mathring{D}_j \to G$ as usual, which are compatible with the action of the finite "isotropy groups" $G_i \subseteq G$ on $\mathring{D}_i^2 \times F$. A good reference for the general theory is [Ho]; in our case $F = S^1$ and $G = Top(S^1)$. The reduction of the structural group to some closed subgroup H of G is possible via the construction of a section of the associated Seifert bundle with fibre $G/_H$ – here, since the inclusion $i : O(2) \hookrightarrow Top(S^1)$ is well-known to be a homotopy equivalence, M^3 can be described as a bundle with O(2) as a structural group. Note that Seifert's condition (b) implies that each finite isotropy group G_i is already contained in O(2).

The classification theorem extends to cover this wider class of examples. The only new ingredient is that O(2) contains reflections of the generic fibre, hence along some non-trivial curve in M* the fibre may reverse its orientation. As in other contexts this phenomenon is described by a homomorphism $w : \pi_1 M^* \to Z/_2$. Among others we now obtain the subclass n_2 : M* is non-orientable, all generators of

$\pi_1 M^*$ reverse orientation and hence the total space M is orientable. Up to O(2)- diffeomorphism a manifold with $\varepsilon = n_2$ is specified by the orbit invariants

$$\{b\ ;\ (n_2, g); (\alpha^1, \beta_1) \ldots (\alpha^r, \beta_r)\}\ , \quad b \in \mathbb{Z},\ 0 < \beta_j < \alpha_j, (\alpha_j, \beta_j) = 1.$$

It is important to understand this subclass, because it plays an important role in Chapters III and IV below.

Fundamental groups. In terms of the notation already introduced $\pi_1(M)$ has a presentation as follows:

Generators: $h,\ q_o,\ q_1 \ldots q_r \begin{cases} a_1\ b_1 \ldots a_g b_g & M^* \text{ orientable} \\ v_1 \ldots v_g & M^* \text{ non-orientable.} \end{cases}$

Relations: $\left.\begin{array}{l} a_i h a_i^{-1} = h \\ b_i h\ b_i^{-1} = h \end{array}\right\} \quad (o_1)$

$v_i h v_i^{-1} = h^{\eta} \quad \begin{cases} \eta(n_1) = 1, & \text{M non-orientable} \\ \eta(n_2) = 1, & \text{M orientable} \end{cases}\ ,$

$q_j h = h q_j,\ j = 1, \ldots, r.$

(these are the commuting relations.)

$q_j^{\alpha_j} = h^{\beta_j} = 1$

(Geometrically the meridian m equals $\alpha q + \beta h$.)

$q_o(q_1 \ldots q_r [a_1, b_1] \ldots [a_g, b_g]) = 1$ or

$q_o(q_1 \ldots q_r\ v_1^2\ v_1^2 \ldots v_g^2) = 1$

$q_o h^b = 1.$