LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

The books in the series listed below are available from booksellers, or, in case of difficulty, from Cambridge University Press.

34 Representation theory of Lie groups, M.F. ATIYAH *et al*
36 Homological group theory, C.T.C. WALL (ed)
39 Affine sets and affine groups, D.G. NORTHCOTT
40 Introduction to H_p spaces, P.J. KOOSIS
43 Graphs, codes and designs, P.J. CAMERON & J.H. VAN LINT
45 Recursion theory: its generalisations and applications, F.R. DRAKE & S.S. WAINER (eds)
46 p-adic analysis: a short course on recent work, N. KOBLITZ
49 Finite geometries and designs, P. CAMERON, J.W.P. HIRSCHFELD & D.R. HUGHES (eds)
50 Commutator calculus and groups of homotopy classes, H.J. BAUES
51 Synthetic differential geometry, A. KOCK
57 Techniques of geometric topology, R.A. FENN
59 Applicable differential geometry, M. CRAMPIN & F.A.E. PIRANI
62 Economics for mathematicians, J.W.S. CASSELS
66 Several complex variables and complex manifolds II, M.J. FIELD
69 Representation theory, I.M. GELFAND *et al*
74 Symmetric designs: an algebraic approach, E.S. LANDER
76 Spectral theory of linear differential operators and comparison algebras, H.O. CORDES
77 Isolated singular points on complete intersections, E.J.N. LOOIJENGA
78 A primer on Riemann surfaces, A.F. BEARDON
79 Probability, statistics and analysis, J.F.C. KINGMAN & G.E.H. REUTER (eds)
80 Introduction to the representation theory of compact and locally compact groups, A. ROBERT
81 Skew fields, P.K. DRAXL
82 Surveys in combinatorics, E.K. LLOYD (ed)
83 Homogeneous structures on Riemannian manifolds, F. TRICERRI & L. VANHECKE
86 Topological topics, I.M. JAMES (ed)
87 Surveys in set theory, A.R.D. MATHIAS (ed)
88 FPF ring theory, C. FAITH & S. PAGE
89 An F-space sampler, N.J. KALTON, N.T. PECK & J.W. ROBERTS
90 Polytopes and symmetry, S.A. ROBERTSON
91 Classgroups of group rings, M.J. TAYLOR
92 Representation of rings over skew fields, A.H. SCHOFIELD
93 Aspects of topology, I.M. JAMES & E.H. KRONHEIMER (eds)
94 Representations of general linear groups, G.D. JAMES
95 Low-dimensional topology 1982, R.A. FENN (ed)
96 Diophantine equations over function fields, R.C. MASON
97 Varieties of constructive mathematics, D.S. BRIDGES & F. RICHMAN
98 Localization in Noetherian rings, A.V. JATEGAONKAR
99 Methods of differential geometry in algebraic topology, M. KAROUBI & C. LERUSTE
100 Stopping time techniques for analysts and probabilists, L. EGGHE

101 Groups and geometry, ROGER C. LYNDON
103 Surveys in combinatorics 1985, I. ANDERSON (ed)
104 Elliptic structures on 3-manifolds, C.B. THOMAS
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
106 Syzygies, E.G. EVANS & P. GRIFFITH
107 Compactification of Siegel moduli schemes, C-L. CHAI
108 Some topics in graph theory, H.P. YAP
109 Diophantine Analysis, J. LOXTON & A. VAN DER POORTEN (eds)
110 An introduction to surreal numbers, H. GONSHOR
111 Analytical and geometric aspects of hyperbolic space, D.B.A.EPSTEIN (ed)
112 Low-dimensional topology and Kleinian groups, D.B.A. EPSTEIN (ed)
113 Lectures on the asymptotic theory of ideals, D. REES
114 Lectures on Bochner-Riesz means, K.M. DAVIS & Y-C. CHANG
115 An introduction to independence for analysts, H.G. DALES & W.H. WOODIN
116 Representations of algebras, P.J. WEBB (ed)
117 Homotopy theory, E. REES & J.D.S. JONES (eds)
118 Skew linear groups, M. SHIRVANI & B. WEHRFRITZ
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
122 Non-classical continuum mechanics, R.J. KNOPS & A.A. LACEY (eds)
124 Lie groupoids and Lie algebroids in differential geometry, K. MACKENZIE
125 Commutator theory for congruence modular varieties, R. FREESE & R. MCKENZIE
126 Van der Corput's method for exponential sums, S.W. GRAHAM & G. KOLESNIK
127 New directions in dynamical systems, T.J. BEDFORD & J.W. SWIFT (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
129 The subgroup structure of the finite classical groups, P.B. KLEIDMAN & M.W.LIEBECK
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
132 Whitehead groups of finite groups, ROBERT OLIVER
133 Linear algebraic monoids, MOHAN S. PUTCHA
134 Number theory and dynamical systems, M. DODSON & J. VICKERS (eds)
135 Operator algebras and applications, 1, D. EVANS & M. TAKESAKI (eds)
136 Operator algebras and applications, 2, D. EVANS & M. TAKESAKI (eds)
137 Analysis at Urbana, I, E. BERKSON, T. PECK, & J. UHL (eds)
138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR and A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
142 The geometry of jet bundles, D.J. SAUNDERS
143 The ergodic theory of discrete groups, PETER J. NICHOLLS
144 Introduction to uniform spaces, I.M. JAMES
145 Homological questions in local algebra, JAN R. STROOKER
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
147 Continuous and discrete modules, S.H. MOHAMED & B.J. MÜLLER
148 Helices and vector bundles, A.N. RUDAKOV et al
152 Oligomorphic permutation groups, P. CAMERON
154 Number theory and cryptography, J. LOXTON (ed)
156 Twistors in mathematics and physics, T.N. BAILEY & R.J. BASTON (eds)

London Mathematical Society Lecture Note Series. 145

Homological Questions in Local Algebra

Jan R. Strooker
Professor of Algebra, University of Utrecht

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester Melbourne Sydney

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521315265

First published 1990
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-31526-5 paperback

CONTENTS

PREFACE

It all began with Serre's beautiful 1957/58 course at the Collège de France on Algèbre locale - Multiplicités [Se]. Here he introduced a local algebraic version of the geometric notion of intersection multiplicity as follows. Let A be a d-dimensional regular local ring and M and N finitely generated A-modules such that $M \otimes_A N$ has finite length. Then he defined their intersection multiplicity $\chi(M,N)$ as $\Sigma_{j=1}^{d}(-1)^j \ell(\mathrm{Tor}_j^A(M,N))$ and the course culminated in the three statements

(M_0) $\dim M + \dim N \leq d$;

(M_1) In case of an inequality above, $\chi(M,N) = 0$;

(M_2) In case there is equality, $\chi(M,N) > 0$.

Serre proved (M_0) in general and (M_1) and (M_2) except for ramified local rings in mixed characteristic. He then cagily added: "Il est naturel de conjecturer que ces résultats sont vrais pour tous les anneaux réguliers.", thus raising the first of the homological questions of the type this book addresses. Almost 30 years later, (M_1) was proved by P. Roberts [Ro 85], [Ro 87b] and by Gillet-Soulé [GS] using different methods, but (M_2) remains open in the ramified case.

That the (M_i) are not true for an arbitrary noetherian local ring is seen in the example of a 3-dimensional local domain $A = k[[X,Y,U,V]]/(XY-UV)$, k a field, and its 2-dimensional modules

M = A/(X,U), N = A/(Y,V). Since $M\otimes_A N$ has length 1, statement (M_0) is violated.

It was then surmised that the operative fact for regular rings - which in those days had just been established by Auslander-Buchsbaum and by Serre himself - is that every module has finite projective dimension. To exclude examples as above, the (M_i) were therefore conjectured for an arbitrary noetherian local ring provided M or perhaps M and N possessed finite projective dimension. This was the point of view taken by M. Auslander, which stimulated a lot of research. It therefore created quite a stir when a module M of finite length and projective dimension was constructed over the 3-dimensional ring A above, such that for our module N the intersection multiplicity $\chi(M,N) = -1$, though dim M + dim N = 2 < 3, contradicting a generalization of (M_i), i = 1,2 with only M of finite projection dimension [DHM]. Over this (and any) complete intersection ring A however, (M_1) is true if both M and N have finite projective dimension, [Ro 85], [Ro 87b], [GS].

In this book I shall concentrate on such a generalization of (M_0) or rather a weaker form as explained in section 8.5. This Intersection Conjecture was proved by Peskine-Szpiro in their pioneering joint thesis [PS 73] in positive characteristic and for rings essentially of finite type over a field of characteristic 0, a restriction which was soon removed by Hochster [Ho 75a]. The statement is now the Intersection Theorem 8.4.4, since P. Roberts recently managed to prove the mixed characteristic case [Ro 87a], [Ro 89]. Fittingly perhaps, his main tool is the theory of local Chern characters and their Riemann-Roch Theorem, which is part of the Fulton-MacPherson treatment of intersection multiplicities in algebraic geometry [Fu], bringing the development back full circle to the underlying geometry. Of this part of Roberts' proof I can only give a brief indication in section 13.1.

The Intersection Theorem with related results and conjectures is a focal point in the book. I also treat the Direct Summand Conjecture and the Monomial Conjecture which are equivalent and prove them in equal characteristic, Theorem 10.3.5, along the general lines of [Ho 73b]. Hochster has shown in a long and rich paper [Ho 83] that the latter, ostensibly not overly homological, yet implies the Intersection Theorem and its consequences. But since the Monomial Conjecture is still open in mixed characteristic, I have not taken this route.

There are two main strategies for attacking these homological questions. The first deploys a range of subtle homological arguments, as in [PS 73], [Ro 76], [Ro 87a]. Most spectacular in the second approach is the brute force construction by Hochster of Big Cohen-Macaulay modules [Ho 75a]. Both methods have in common that they work in positive characteristic by exploiting the Frobenius endomorphism, and then obtain the result in equal characteristic zero by general methods, as explained in Chapter 12. There is however a notable exception, Roberts' proof of New Intersection over the complex field [Ro 80c].

In this book I combine the best of both worlds, plugging our Big Cohen-Macaulay module into the Acyclicity Lemma to obtain New Intersection, as first shown by Foxby [Fo 77b]. I present however a refinement of Hochster's original construction of Big Cohen-Macaulay modules which goes back to [BS] and [Ba].

I do not list all the homological conjectures, nor trace their often surprising interconnections. This has been done in [PS 73] and repeatedly and authoritatively by Hochster [Ho 73a], [Ho 75a], [Ho 75b], [Ho 78], [Ho 79], [Ho 82], [Ho 87]. To some extent I tried my hand at this in [St 85], which may serve as a synopsis of the main thrust of this book.

In writing a monograph of this kind, a problem is where to start. I have decided to take standard homological algebra and the theory of homological dimension for granted, but in Chapter 3 rather carefully treat injective envelopes and Matlis duality. While commutative algebra is assumed at the level of, say, [AM], completion is discussed in Chapter 2 and regular sequences and parameters in Chapters 5 and 8 respectively. In this way I hope that the book will not only be of interest to commutative algebraists, but that mathematicians in other areas and also persevering graduate students, will be able to partake of these fascinating homological questions.

Now for acknowledgements. First I should like to salute Mel Hochster, whose unflagging enthusiasm for the homological conjectures drew me into the subject. This book originated in a course and seminar which I taught more years ago than I care to remember. Several participants have greatly assisted in the writing of the book. Lex Vermeulen and Ton Vorst contributed drafts and outlines of sections in the early stages, and so did Dick Buijs. Jaap Bartijn deserves a special word of thanks. Quite a bit of material in the book derives from our collaboration on [BS] or from his thesis [Ba]. Moreover, we have often discussed the contents of the book and he has written drafts for certain chapters. Dick Buijs has throughout been a great help in reading innumerable versions of chapters. His pertinent comments have reduced the number of errors and improved readability. This also holds for Anne-Marie Simon who performed the same task during the later stages.

For a friendly gesture of a different nature I am indebted to Lou van den Dries. He offered to write a chapter where Big Cohen-Macaulay modules are shown to exist in equal characteristic 0 once they are known to exist in all finite characteristics. The argument really pertains in greater generality, and Van den Dries has given us his view of the matter in Chapter 12, which differs in certain respects from previous treatments. This chapter also offers a valuable introduction, at fairly elementary level, to Artin

Approximation and henselization. Since this material is almost entirely independent from the rest of the book, and was written several years ago, Utrecht University put out a tract and distributed a limited number of copies in 1983. Van den Dries would like to thank Craig Huneke for spotting a gap in the proof of Approximation in that early rendering.

"De Pauwhof" in Wassenaar deserves thanks for several sojourns of sustained work. The publisher, lastly incarnated in David Tranah, has shown courtesy and patience throughout. Finally, it is a pleasure to thank Karin Berlang for cheerfully and competently processing a hundred visions and revisions before the taking of a toast and tea.

Utrecht, Jan R. Strooker

December 1989