1 HOMOLOGICAL PRELIMINARIES

In writing a book on an advanced topic one cannot start from scratch, so one needs to make a choice of basic subjects which the reader is supposed to be familiar with. Here we assume a working knowledge of homological algebra and the theory of various homological dimensions, as expounded for instance in standard texts like [CE], [No 60], [HSa] or [Rot].

In this brief chapter we merely explain a few technical devices which are less uniformly known. They concern the exactness of certain complexes of modules. In the first section we deal with abstract Acyclicity Lemma's for single complexes in abelian categories, which will i.a. be used to deduce the parent version due to Peskine-Szpiro, 6.1.1. We also restate uniqueness of derived functors in a form suitable to our purposes. In 1.2 we consider tensor products of complexes of modules, preparing for certain results in Chapter 7. Most readers will hardly feel tempted to study this material for its own sake, but may like to refer back when it is needed in these later chapters.

1.1 ACYCLICITY LEMMA'S

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A sequence F^i, $i = 0,1,2,\dots$, of covariant additive functors from $\mathscr{C}$ to $\mathscr{D}$ is called right connected exact if to each short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathscr{C}$ there corresponds a functorial long exact sequence

$$0 \to F^0C' \to F^0C \to F^0C'' \to F^1C' \to \dots \to F^iC' \to F^iC \to F^iC'' \to F^{i+1}C' \to \dots \text{ in } \mathscr{D}.$$

For each object C in $\mathscr{C}$ we put $f^-C = \inf\{i \mid F^iC \neq 0\}$. Notice that f^-C is either ∞ or a nonnegative integer.

1.1.1 PROPOSITION. Let F^i be a right connected exact sequence of functors from $\mathscr{C}$ to $\mathscr{D}$. If $C_\bullet = 0 \to C_s \xrightarrow{d_s} C_{s-1} \to \dots \xrightarrow{d_1} C_0$ is a complex in $\mathscr{C}$ such that for $i = 1,\dots,s$ both $f^-C_i \geq i$ and either $H_i(C_\bullet) = 0$ or $f^-H_i(C_\bullet) = 0$, then $C_\bullet$ is exact.

PROOF. Writing H_i for the homology $H_i(C_\bullet)$, we shall successively show that $H_s = 0$, $H_{s-1} = 0$, ... , $H_1 = 0$. If $H_s \neq 0$, then $F^0H_s \neq 0$, and the monomorphism $H_s \to C_s$ yields $F^0C_s \neq 0$ which contradicts the fact that $f^-C_s \geq s > 0$.

For each i we shall write $T_i = \operatorname{coker}(d_{i+1})$ and $K_i = \ker(d_i)$. Then $T_s = C_s$ and $f^-T_s \geq s$, so that, for $0 < r < s$, our induction hypothesis may read: $H_i = 0$ and $f^-T_i \geq i$ for $0 < r < i \leq s$.

To take the induction step, consider the two sequences $0 \to T_{r+1} \to C_r \to T_r \to 0$ and $0 \to T_{r+1} \to K_r \to H_r \to 0$ which are exact because $H_{r+1} = 0$. The first one gives rise to an exact sequence $F^jC_r \to F^jT_r \to F^{j+1}T_{r+1}$ for each $j \geq 0$. Since $f^-T_{r+1} \geq r+1$ and $f^-C_r \geq r$, we find $f^-T_r \geq r$. The second one induces an exact sequence $0 \to F^0T_{r+1} \to F^0K_r \to F^0H_r \to F^1T_{r+1} \to \dots$ in $\mathscr{D}$. Since $r \geq 1$, we know that $F^0T_{r+1} = F^1T_{r+1} = 0$. If $H_r \neq 0$, then $F^0H_r \neq 0$ and thus $F^0K_r \neq 0$. But then $F^0C_r \neq 0$, which contradicts our assumptions.

One of the advantages of such an abstract formulation in terms of abelian categories, first noticed by A.-M. Simon [Si], [St 90], is that we right away have a dual statement. Terminology and notation should be self-explanatory.

1.1.2 PROPOSITION. Let F_i be a left connected exact sequence of functors between $\mathcal{C}$ and $\mathcal{D}$. If $C^\bullet = C^0 \to C^1 \to \dots \to C^s \to 0$ is a complex in $\mathcal{C}$ such that, for $i = 1,\dots,s$, both $f\_C^i \geq i$ and either $H^i(C^\bullet) = 0$ or $f\_H^i(C^\bullet) = 0$, then $C^\bullet$ is exact.

It is left to the reader to formulate companion versions for contravariant functors. To prepare for our next proposition, suppose that $F = (F^i)$ and $G = (G^i)$ are both right connected exact sequences of functors from $\mathcal{C}$ to $\mathcal{D}$. A morphism $\sigma\colon F \to G$ is a collection of morphisms $\sigma^i\colon F^i \to G^i$, $i \geq 0$, such that for every short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} F^iC'' & \longrightarrow & F^{i+1}C' \\ {\scriptstyle \sigma^i(C'')}\downarrow & & \downarrow{\scriptstyle \sigma^{i+1}(C')} \\ G^iC'' & \longrightarrow & G^{i+1}C' \end{array}$$

commutes in $\mathcal{D}$. The result we shall need does not occur in this form in the standard texts, but is an immediate consequence of their treatment of satellites and derived functors. A more general version is proved in [St 78, Th. 3.4.3].

1.1.3 PROPOSITION. Let $F = (F^i)$ and $G = (G^i)$ both be right connected exact sequences of covariant functors between abelian categories $\mathscr{C}$ and $\mathscr{D}$. Suppose there is an isomorphism σ^0: $F^0 \to G^0$. This extends to an isomorphism σ: $F \to G$ provided $F^iI = G^iI = 0$ for every $i \geq 1$ and every injective object I in $\mathscr{C}$.

We shall also have occasion to use two dual versions, where the higher functors are required to vanish on projectives.

1.2 A FEW ISOMORPHISMS OF COMPLEXES

We shall need several facts on complexes, which are of a fairly simple nature, because key results concern complexes of vector spaces. Ad hoc proofs could certainly be provided. On the other hand, this material is rightfully part of the theory of derived categories, to which there exists several introductions in [Ha], [De], [Bor] and [Iv].

We shall strike a compromise, since fortunately the basic techniques which we shall use are treated carefully and in detail in the treatise [Bo 80]. In this section therefore we adhere to notation and terminology of Bourbaki and often refer to this work for the proofs. Our treatment is in the spirit of [Fo 77a] and [Fo 79].

We take a fixed (commutative) ring A and work with complexes $C^\bullet$ and $C_\bullet$ of A-modules. Here $C^\bullet$ will always stand for an ascending complex $\ldots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \ldots$ with $d^i \circ d^{i-1} = 0$ and $C_\bullet$ for a descending complex with d_i: $C_i \to C_{i-1}$ and $d_i \circ d_{i+1} = 0$. It is sometimes convenient to switch from one to the other by writing $C_i = C^{-i}$ and $d_i = d^{-i}$.

A morphism (of degree 0) $u: C^\bullet \to D^\bullet$ is called a homologism between these complexes if it induces an isomorphism $H(u): H(C^\bullet) \simeq H(D^\bullet)$ [Bo 80, §2, Déf. 3]. Most authors speak of a quasi-isomorphism or quism, but Bourbaki's term is more descriptive. Standard examples of homologisms are provided by resolutions. If C is a module, one can regard it as a complex $C^\bullet$ by putting $C^0 = C$ and $C^i = 0$ for all $i \neq 0$. If $0 \to C \xrightarrow{\epsilon} E^0 \to E^1 \to \ldots$ is for instance an injective resolution of C, and $E^\bullet$ is the complex $0 \to E^0 \to E^1 \to \ldots$, a homologism $u: C^\bullet \to E^\bullet$ is defined by $u = (u^i)$ with $u^0 = \epsilon$ and $u^i = 0$ for $i \neq 0$.

Given two complexes $C_\bullet$ and $D_\bullet$, we have the notion of the tensor product complex $C_\bullet \otimes_A D_\bullet$ [Bo 80, §4.1]. It is a descending complex with chain modules $(C_\bullet \otimes_A D_\bullet)_n = \oplus_{i+j=n} C_i \otimes_A D_j$. For the differential distinct conventions are in use, one of which is chosen by Bourbaki. We shall not make this explicit, since the reader is referred back for the proofs anyway. Similarly for $C_\bullet$, and $D^\bullet$ an ascending complex, $\mathrm{Homgr}_A(C_\bullet, D^\bullet)$ is defined [Bo 80, §5.1]. In what follows we shall only need this when $C_\bullet$ is concentrated in degree 0, in other words may be thought of as a module $C = C_0$, and it is easy to see that then $\mathrm{Homgr}_A(C_\bullet, D^\bullet) \simeq \mathrm{Hom}_A(C, D^\bullet)$ so that we may forget about the fancier notion.

1.2.1 PROPOSITION. Let $u: C^\bullet \to D^\bullet$ be a homologism of complexes and $F^\bullet$ a complex of flat modules with $F^i = 0$ for i sufficiently large. Then $1_{F^\bullet} \otimes u: F^\bullet \otimes_A C^\bullet \to F^\bullet \otimes_A D^\bullet$ is a homologism.

PROOF. This is [Bo 80, §4, Prop. 4].

1.2.2 PROPOSITION. Let A be a noetherian ring and C a finitely generated A-module. Suppose $F^\bullet$ is a complex of flat modules and $E^\bullet$ a complex of injective modules. Then there is an isomorphism of complexes

$F^\bullet \otimes_A \mathrm{Hom}_A(C,E^\bullet) \simeq \mathrm{Hom}_A(C,F^\bullet \otimes_A E^\bullet)$ and the complex $F^\bullet \otimes_A E^\bullet$ consists of injectives.

PROOF. For each pair of indices i and j, there is a natural homomorphism from $F^i \otimes_A \mathrm{Hom}_A(C,E^j)$ to $\mathrm{Hom}_A(C,F^i \otimes_A E^j)$ which is an isomorphism because C is finitely presented and F^i is flat [Bo 61a, Ch. 1, §2, Prop. 10]. It is straightforward to show that these maps commute with the differentials, hence yield the desired isomorphism of complexes. In fact, the compatibility is discussed in the greater generality of Homgr in [Bo 80, §5, 7, (16)].

For each i and j the isomorphism above extends to isomorphisms $F^i \otimes_A \mathrm{Ext}^n_A(C,E^j) \simeq \mathrm{Ext}^n_A(C,F^i \otimes_A E^j)$, $n \geq 0$, by [Bo 80, §6, Prop. 7c)] and, since $\mathrm{Ext}^n_A(C,E^j) = 0$ for $n \geq 1$, so is $\mathrm{Ext}^n_A(C,F^i \otimes_A E^j)$. Now it is enough to test injectivity for cyclic modules $C = A/\mathfrak{a}$, $\mathfrak{a}$ an ideal in A, so the $F^i \otimes_A E^j$ are all injective. Since over a noetherian ring every direct sum of injective modules is injective, all chain modules in the complex $F^\bullet \otimes_A E^\bullet$ are injective. For the two facts on injective modules which we have just used, the reader may for instance consult section 3.1 of this book.

1.2.3 PROPOSITION. Let $V_\bullet$ and $W_\bullet$ be complexes of vector spaces over a field k and suppose that either $V_i = 0$ for i sufficiently small or $W_j = 0$ for j sufficiently small. Then $H(V_\bullet) \otimes_k H(W_\bullet) \simeq H(V_\bullet \otimes_k W_\bullet)$. In other words, for each pair of indices i and j, there is an isomorphism of vector spaces $H_n(V_\bullet \otimes_k W_\bullet) \simeq \oplus_{i+j=n} H_i(V_\bullet) \otimes_k H_j(W_\bullet)$.

PROOF. Since k-vector spaces are certainly k-flat, this is a special case of the Künneth formula [Bo 80, §4, Th. 3, Cor. 4].

As in the rest of this book, (A,m,k) will be standard notation for a local ring A with maximal ideal $\mathfrak{m}$ and residue class field $k = A/\mathfrak{m}$. The next theorem is taken from [Ba, Ch. IV, Prop. 2.7], see also [BS, Th. 4.1].

1.2.4 THEOREM. Let $(A,\mathfrak{m},k)$ be a noetherian local ring and $F^\bullet = 0 \to F^{-s} \to \dots \to F^0 \to 0$ a finite complex of flat A-modules. Let C be an A-module, $E^\bullet$ an injective resolution of C and $I^\bullet$ the complex $F^\bullet \otimes_A E^\bullet$. Then $I^\bullet$ is a complex of injectives with $I^n = 0$ for $n < -s$, and $H^n(I^\bullet) \simeq H^n(F^\bullet \otimes_A C)$. Furthermore $H^n(\mathrm{Hom}_A(k,I^\bullet)) \simeq \oplus_{i+j=n} H^i(F^\bullet \otimes_A k) \otimes_k \mathrm{Ext}^j_A(k,C)$.

PROOF. We have seen in 1.2.2 that $I^\bullet$ is a complex of injectives, while $I^n = 0$ for $n < -s$ by definition of the tensor product of complexes. The homologism $C \to E^\bullet$ is preserved by $1_{F^\bullet} \otimes_A -$ according to 1.2.1, so $H^n(I^\bullet) \simeq H^n(F^\bullet \otimes_A C)$.

Next observe that there is an isomorphism of complexes $F^\bullet \otimes_A \mathrm{Hom}_A(k,E^\bullet) \simeq \mathrm{Hom}_A(k,I^\bullet)$ by 1.2.2. Now $\mathrm{Hom}_A(k,E^\bullet)$ is a complex of k-vector spaces, so $F^\bullet \otimes_A \mathrm{Hom}_A(k,E^\bullet) \simeq F^\bullet \otimes_A k \otimes_k \mathrm{Hom}_A(k,E^\bullet)$. We invoke 1.2.3 with $V^\bullet = F^\bullet \otimes_A k$ and $W^\bullet = \mathrm{Hom}_A(k,E^\bullet)$, since that isomorphism could equally well have been stated for ascending complexes - here $V^i = 0$ for $i > 0$. Observing that $H^j(W^\bullet) \simeq \mathrm{Ext}^j_A(k,C)$, we obtain the final isomorphism of the theorem.

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2. ADIC TOPLOGIES AND COMPLETIONS

The homological conjectures which are discussed in this book are local and, what is more, need only be proved over a complete noetherian local ring. The structure of such rings, expressed in the Cohen Structure Theorems, is heavily used in their solution. Apart from these, we only need results from this chapter incidentally.

There are in the literature several excellent accounts of the topics in the title [AM], [Bo 61b, Ch. 3], [ZS], [Ma 86], but these do tend to concentrate all too soon on finitely generated modules over noetherian rings. Nevertheless, there exists a quite attractive theory at least for arbitrary modules over noetherian rings, and in some cases we may even waive the noetherian condition. Since this book features the "construction" of certain infinitely generated complete modules with good properties, cf. Theorems 5.2.3 and 9.1.1, and complete modules are drawing an increasing measure of attention [Ba], [Si], we have chosen to present an outline of this theory. In doing so we recall several standard results without proof, and for the less standard ones give either a proof, hints for a proof or a reference.

In the first section we show how the notion of purity can serve in the realm of adic topologies when the Artin-Rees Lemma is unavailable. On the other hand, a pure submodule is a poor man's direct summand, and this paves the way to our proof of Hochster's Direct Summand Theorem in equal characteristic, 10.3.5.

The second section discusses completion, briefly yet in somewhat more detail than we shall need.

In the third and final section then, Hensel's Lemma quickly leads to a proof that an equicharacteristic complete noetherian local ring contains a field of representatives.

2.1 INDUCED TOPOLOGIES AND PURITY

Let $\mathfrak{a}$ be an ideal in the ring A and M an A-module. The descending set of ideals $\mathfrak{a}^t$, $t \geq 0$, is taken as a basis of open neighbourhoods of the 0-element of A. This defines the $\mathfrak{a}$-adic topology of A which then is a topological ring. The module M becomes a topological module over A by taking the submodules $\mathfrak{a}^tM$ to form a basis of open neighbourhoods of 0 in M. To each $m \in M$ we can attach $v(m) = \sup\{t \mid m \in \mathfrak{a}^tM\}$. The values v(m) are nonnegative integers or ∞, and we can define $d(m,n) = 2^{-v(m-n)}$ for m and n in M. This function d is a pseudometric on the topological module M, which is a metric precisely when $\cap_{t=0}^{\infty} \mathfrak{a}^tM = 0$, or in other words, when the module M is Hausdorff in its $\mathfrak{a}$-adic topology. It is well known that $\mathfrak{a}$-adic completions, defined by an inverse limit construction in the next section, can also be described in terms of Cauchy sequences with respect to the pseudometric d, and sometimes we take this point of view.

A chain (M_t): $M = M_0 \supset M_1 \supset \ldots \supset M_t \supset \ldots$ of submodules is called a filtration of M. It is an $\mathfrak{a}$-filtration if $\mathfrak{a}M_t \subset M_{t+1}$ for all t, and a stable $\mathfrak{a}$-filtration if moreover $\mathfrak{a}M_t = M_{t+1}$ for all sufficiently large t. For instance, $(\mathfrak{a}^tM)$ is a stable $\mathfrak{a}$-filtration of M.

An $\mathfrak{a}$-filtration makes M into a topological A-module, and the topology defined by a stable $\mathfrak{a}$-filtration is just the $\mathfrak{a}$-adic topology of M. As is frequently the case, one of the fundamental results in the theory is humbly known as a lemma, the Artin-Rees Lemma:

2.1.1 THEOREM. Let $\mathfrak{a}$ be an ideal in a noetherian ring A and M a finitely generated A-module, with submodule N. There exists an integer $s \geq 0$ such that $\mathfrak{a}^t M \cap N = \mathfrak{a}^{t-s}(\mathfrak{a}^s M \cap N)$ for all $t \geq s$. In particular, the $\mathfrak{a}$-adic topology of N coincides with the topology induced by the $\mathfrak{a}$-adic topology of M.

To see that finite generation of M is essential, consider the example $A = \mathbb{Z}_{(p)}$, p a prime, $\mathfrak{a} = (p)$, $M = \mathbb{Q}$, $A = N \subset M$. Then the $\mathfrak{a}$-adic topology on M is the indiscrete topology hence far from Hausdorff. Thus the induced topology on N is also indiscrete and non Hausdorff. However, the $\mathfrak{a}$-adic toplogy on N is clearly Hausdorff, which also follows from the more general

2.1.2 PROPOSITION. Let $\mathfrak{a}$ be a proper ideal in a noetherian domain A. Then $\cap_{t=0}^{\infty} \mathfrak{a}^t = 0$, so A is Hausdorff in its $\mathfrak{a}$-adic topology.

This is a special case of the next result, Krull's celebrated Intersection Theorem. Though Krull's work is older, a particularly efficient proof using 2.1.1 is given in [AM, Th. 10.17].

2.1.3 THEOREM. Let $\mathfrak{a}$ be an ideal in a noetherian ring A and M a finitely generated A-module equipped with the $\mathfrak{a}$-adic topology. Then the closure of 0 in M is $\cap_{t=0}^{\infty} \mathfrak{a}^t M$ and consists of those $m \in M$ which are