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Combinatorics and Ramanujan's "Lost" Notebook

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1. Introduction.

L. J. Rogers' paper (Rogers; 1894) which contains the Rogers-Ramanujan identities together with their proof was ignored for 20 years before Ramanujan came across it while leafing through old volumes of the Proceedings of the London Mathematical Society. In the interim, Ramanujan had discovered the Rogers-Ramanujan identities empirically, and they were making the rounds as major unsolved problems (cf. Hardy; 1940, p. 91). This is undoubtedly one of the very few times that a set of significant unsolved problems was solved 20 years before it was posed.

The most obvious reason Rogers' paper lay buried is that it is page after page of q-series identities with the Rogers-Ramanujan identities sneaking past in mild disguise on page 10 of this tour de force.

As more discoveries were made, the subject became even less readable. The Rev. F. H. Jackson was one of the early pioneer q-series researchers. His papers also read much like Rogers'. It is not surprising to read in Jackson's obituary (Chaundy (1962)); "Once (with a whimisical smile one imagines) he [Jackson] recounted the occasion of his quarrel with our Society [the L.M.S.]: he had read a paper when someone remarked: 'Surely, Mr. President, we have heard all this before.' He strode from the room and never darkened our pages again." As it turned out this critical remark was directed at what was, in fact, Jackson's most valuable paper. Again the result was one equation among many which were indistinguishable to the outsider.

These incidents point up one of the main difficulties in presenting results on q-series to a wide audience. While there have been interesting interactions of this subject with physics (Baxter; 1980, 1982 or Andrews et al.; 1984), transcendental number theory (Richmond and Szekeres; 1981), group theory (Lusztig; 1977 or Andrews; 1977, 1984), and additive number theory (Andrews; 1972), nonetheless a paper like Slater's compendium of Rogers-Ramanujan type identities (Slater; 1952) leaves the impression that it is impossible to have any idea of what is really going on.

With this background we turn to Ramanujan's "Lost" Notebook (cf. Andrews; 1979 or Rankin; 1982). This document contains over 600 unproved results of which at least two thirds are q-series identities. Again the superficial sameness of these results leaves one daunted.

Having thus criticized some very fine mathematicians, I hope I will be forgiven if I fail to provide the Olympian overview which you are probably expecting. I do hope to suggest an approach to increasing understanding. In particular I want to describe some means by which one might hope to gain insight about a series like

$$(1.1)\qquad \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(1-q)(1-q^3)\cdots(1-q^{2n-1})}{(1+q^2)^2(1+q^4)^2\cdots(1+q^{2n})^2}$$

for example. I choose this example because it appears prominently in Ramanujan's "Lost" Notebook, and I devoted a lengthy paper to a study of it and several related series (Andrews; 1981a). Indeed I gave the first 36 coefficients of its Maclaurin series expansion (Andrews; 1981a, pp. 44-45). As is clear from my comments in Section 5 of that paper, I hadn't the least notion of any reasonable combinatorial significance of this series. At least the methods I describe herein will easily yield a straightforward combinatorial interpretation of (1.1) (cf. Section 7).

Section 2 of this paper will provide a general setting for series of this type. In Section 3, we shall describe some means to fit various q-series into this framework. The remainder of the paper considers applications of these ideas to some of the more incomprehensible q-series. Among the topics covered are all the classes of mock theta functions given in Ramanujan's last letter to Hardy (Ramanujan; 1927). Also in Sections 6 and 7 we examine two classes of functions that also appear in the "Lost" Notebook. We then show how to derive the Rogers-Ramanujan type identities given in Andrews (1981b) for Regime II of Baxter's hard hexagon model.

2. The Combinatorial Setting.

Part of the astonishing nature of Ramanujan's genius lies in his ability to find important formulas and intricate relationships without possessing either a related general theory or even the results in question in full generality (Hardy; 1940, p. 14).

The value of generality in this subject is immensely important. The only proofs of the Rogers-Ramanujan identities with no generality are Schur's incredibly brilliant and intricate combinatorial arguments (Schur; 1917) and the extension of Schur's treatment to a bijective proof (Garsia & Milne; 1981). In contrast, Watson (1929) gives a proof in such great generality that he only needs to base his arguments on the fact that two polynomials of degree n agreeing at $n+1$ values must be identical.

The introduction of some generality is the key here. In effect we shall extend each function of the one variable q to a function of two variables.

Definition 1. A Ramanujan statistic is an ordered pair (ρ,S) where S is a subset of the set of all partitions of all nonnegative integers and ρ is a nonnegative arithmetic function on S with the condition that $\rho^{-1}(n)$ is a finite set for each nonnegative integer n.

Definition 2. The RS-polynomial for the Ramanujan statistic (ρ,S) is

$$p_n(\rho,S|q) = \sum_{\substack{\pi\in S \\ \rho(\pi)\leq n}} q^{\sigma(\pi)}, \tag{2.1}$$

where $\sigma(\pi)$ is the integer partitioned by π.

Definition 3. The RS-generating function for the Ramanujan statistic (ρ,S) is

(2.2) $$F(\rho,S|q,t) = \sum_{n=0}^{\infty} p_n(\rho,S|q)t^n.$$

Definition 4. Let $P(S;n)$ denote the number of partitions of n that lie in S.

Definition 5. Let $D(\rho,S;n)$ denote the number of partitions $\pi(\in S)$ of n for which $\rho(\pi)$ is even minus the number of those for which $\rho(\pi)$ is odd.

Lemma 1. For $|q| < 1$,

(2.3) $$\sum_{n=0}^{\infty} P(S;n)q^n = \lim_{t\to 1^-}(1-t)F(\rho,S|q,t),$$

(2.4) $$\sum_{n=0}^{\infty} D(\rho,S;n)q^n = \lim_{t\to -1^+}(1-t)F(\rho,S|q,t).$$

Proof. We note that by Definition 1,

(2.5) $$\sum_{n\geq 0} P(S;n)q^n = \lim_{n\to\infty} p_n(\rho,S|q)$$
$$= \lim_{t\to 1^-}(1-t)F(\rho,S|q,t),$$

by Abel's lemma (Andrews, 1971, p. 190).

(2.6) $$\sum_{n\geq 0} D(\rho,S;n)q^n = p_0(\rho,S|q) + \sum_{n=1}^{\infty}(-1)^n(p_n(\rho,S|q)-p_{n-1}(\rho,S|q))$$
$$= \lim_{t\to -1^+}(1-t)F(\rho,S|q,t).$$

The convergence in each instance is guaranteed by the fact that the $p_n(\rho,S|q)$ are polynomials with nonnegative coefficients and for each j the coefficient of q^j in $p_n(\rho,S|q)$ is bounded by the total number of partitions of j and is fixed for $n>n_0(j)$ [Definition 1].□

In our applications we shall find that (2.3) and (2.4) are often our starting points. It also happens that $F(\rho,S|q^2,q)$ or $F(\rho,S|q,q)$ arise. I have not been able to interpret nicely these functions in general; however it is possible to do so in many specific applications. Of course, if we define for example

$$\sum_{n=0}^{\infty} T_n(\rho,S;n)q^n = F(\rho,S|q^2,q), \tag{2.7}$$

then clearly $T_n(\rho,S;n)$ is the number of solutions of $\rho(\pi)+2\sigma(\pi)\leq n$, so this at least provides some general combinatorial significance for $F(\rho,S|q^2,q)$. $F(\rho,S|q,q)$ can be similarly interpreted.

3. Determination of Ramanujan Statistics.

We are assuming that the Ramanujan statistic is not provided for us ab initio. Instead we start with series like

$$\sum_{n=0}^{\infty} \frac{(A_1;q^r)_n(A_2;q^r)_n\cdots(A_s;q^r)_n q^{ra\binom{n}{2}} z^n}{(q^r;q^r)_n(B_1;q^r)_n\cdots(B_j;q^r)_n} \tag{3.1}$$

or

$$\sum_{n=0}^{\infty} \frac{(-A_1;q^r)_n(-A_2;q^r)_n\cdots(-A_s;q^r)_n(-1)^{an}z^n q^{ra\binom{n}{2}}}{(-q^r;q^r)_n(-B_1;q^r)_n\cdots(-B_j;q^r)_n} \tag{3.2}$$

or

$$\sum_{n=0}^{\infty} \frac{(A_1q^r;q^{2r})_n(A_2q^r;q^{2r})_n\cdots(A_sq^r;q^{2r})_n q^{ran^2} z^n}{(q^r;q^{2r})_{n+1}(B_1q^r;q^{2r})_n\cdots(B_jq^r;q^{2r})_n} \tag{3.3}$$

where

$$(\alpha;q)_n \equiv (\alpha)_n = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}). \tag{3.4}$$

Often we find in the literature and especially in Ramanujan's work instances of (3.1)-(3.3) intertwined in various identities.

If we define

$$F\begin{pmatrix} A_1,\ldots,A_s;r,a;q,z;t \\ B_1,\ldots,B_j \end{pmatrix} \equiv F(q,t) \tag{3.5}$$

$$= \sum_{n=0}^{\infty} \frac{(A_1t;q^r)_n(A_2t;q^r)_n\cdots(A_st;q^r)_n z^n t^{an} q^{ra\binom{n}{2}}}{(t;q^r)_{n+1}(B_1t;q^r)_n\cdots(B_jt;q^r)_n},$$

then we note that $\lim_{t\to1^-}(1-t)F(q,t)$ is (3.1), $\lim_{t\to-1^+}(1-t)F(q,t)$ is (3.2) and $F(q^2,q^r)$ is (3.3). Thus our $F(q,t)$ fits in with the form of the results given in Lemma 1.

Lemma 2. Let $F(q,t)$ be given by (3.5) with a a positive integer. Then the coefficient of t^n in the expansion of $F(q,t)$ is a polynomial in the A_i, the B_i, z and q. Furthermore $F(q,t)$ satisfies the following simple non-homogeneous q-difference equation:

$$(3.6)\quad (1-t)(1-B_1t)\cdots(1-B_jt)F(q,t)$$
$$= (1-B_1t)(1-B_2t)\cdots(1-B_jt)$$
$$+(1-A_1t)\cdots(1-A_st)t^az\,F(q,tq^r).$$

Proof. The polynomial nature of the coefficients follows immediately from the two classical formulas (Andrews; 1976, p. 36)

$$(3.7)\quad \frac{1}{(A;q)_n} = \sum_{m=0}^{\infty}\begin{bmatrix}n+m-1\\ m\end{bmatrix}A^m,$$

$$(3.8)\quad (A;q)_n = \sum_{m=0}^{n}\begin{bmatrix}n\\ m\end{bmatrix}(-A)^m q^{\binom{m}{2}},$$

where $\begin{bmatrix}r\\ s\end{bmatrix}$ is the q-binomial coefficient or Gaussian polynomial given by

$$(3.9)\quad \begin{bmatrix}r\\ s\end{bmatrix}_q = \begin{bmatrix}r\\ s\end{bmatrix} = \begin{cases}\dfrac{(1-q^r)(1-q^{r-1})\cdots(1-q^{r-s+1})}{(1-q^s)(1-q^{s-1})\cdots(1-q)}, & r,\ s \geqq 0\\ 0 \quad \text{otherwise.}\end{cases}$$

To see (3.6) we observe that

$$(3.10)\quad F(q,t) = \frac{1}{1-t}$$

$$+ \sum_{n=1}^{\infty}\frac{(A_1t;q^r)_n(A_2t;q^r)_n\cdots(A_st;q^r)_n t^{an}z^n q^{ar\binom{n}{2}}}{(t;q^r)_{n+1}(B_1t;q^r)_n\cdots(B_jt;q^r)_n}$$

$$= \frac{1}{1-t} + \frac{(1-A_1t)\dots(1-A_st)t^{a_z}}{(1-t)(1-B_1t)\dots(1-B_jt)} F(q,tq^r),$$

where the last line follows by replacing n by $n+1$ in the sum. Identity (3.6) is merely (3.10) with denominators cleared. □

The steps in applications are now clear

1. Determine $F(q,t)$ from given instances of (3.1)-(3.3).
2. Examine $\lim_{t\to 1^-} (1-t)F(q,t)$ in order, if possible, to identify it as the generating function for partitions lying in some set S. (Often more than one S may turn up).
3. Study the polynomial coefficients for $F(q,t)$ described in Lemma 2. The object is to identify them as RS-polynomials.

Unfortunately I have no general wisdom about Step 3. In practice however there seem to be two possible occurrences: 1) It may be possible to find a simple representation for $F(q,t)$ which transparently yields the relevant Ramanujan statistic. 2) There may be known families of polynomial generating functions related to the set of partitions S found in Step 2; if so it may be possible to identify ρ from these polynomials.

4. The Third Order Mock Theta Function.

We shall now consider two of the principal third order mock theta functions (Watson: 1936, p. 62):

$$(4.1) \qquad f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_n^2},$$

and

$$(4.2) \qquad \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$$

Following the guidance of Section 3, we define

(4.3) $$M\theta_3(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(t)_{n+1}(tq)_n} = F\left(_q;1,2;q,q;t\right),$$

and we note

(4.4) $$f(q) = \lim_{t\to -1^+}(1-t)M\theta_3(q,t),$$

(4.5) $$\omega(q) = M\theta_3(q^2,q)/(1-q).$$

To find our relevant set of partitions we observe that

(4.6) $$\lim_{t\to 1^-} (1-t)M\theta_3(q,t)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2} = \frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n.$$

This last set of identities is often attributed to Euler (Andrews; 1976, p. 21); however, I believe that this q-series first arose in Jacobi's work (Jacobi: 1829, §64). We now see from (4.6) that $S=P$, the set of all partitions of all nonnegative integers.

To determine ρ we apply a transformation to $M\theta_3(q,t)$. Namely

(4.7) $$M\theta_3(q,t) = \frac{1}{1-t} \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(q)_n(x^{-1})_n(x^{-1})_n t^{2n}q^n x^{2n}}{(q)_n(tq)_n(tq)_n}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(tq)_n} .$$

(by Sears; 1951, p. 174, eq. (10.1) with $p=q$, $a=b=x^{-1}$, $e=f=tq$, $c=q$, and $x\to 0$). This last expression is so easy to interpret that we see immediately that the coefficient of t^Mq^N in (4.7) is the number of partitions of N with the largest part plus the number of parts $\leq M$. Hence in this instance

$$\rho(\pi) = gn(\pi),$$

where $gn(\pi)$ denotes the largest part plus the number of parts (the choice of notation refers to the gnomon of the Ferrers graph of π).

Consequently we can now easily identify both $f(q)$ and $(1-q)\omega(q)$ as generating functions of partition functions:

Theorem 1. Let $f(q)$ and $\omega(q)$ be given by (4.1) and (4.2). Then

(4.8) $$f(q) = \sum_{n=0}^{\infty} D(gn,\mathcal{P};n)q^n,$$

and

(4.9) $$(1-q)\omega(q) = \sum_{n=0}^{\infty} \Omega(n)q^n,$$

where $D(gn,\mathcal{P};n)$ is defined in Definition 5 and in this instance is the total number of partitions π of n with $gn(\pi)$ even minus the number with $gn(\pi)$ odd. The partition function $\Omega(n)$ is the number of partitions of n into odd parts where the largest part is at most one more than twice the number of ones.

Remark. The result on $f(q)$ is implicit in a forthcoming book by N. J. Fine (1985).

Proof. Equation (4.8) follows immediately from (2.4), (4.4) and our comments following (4.7). From (4.5), we see that

$$(1-q)\omega(q) = M\Theta_3(q^2,q)$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{(q^3;q^2)_n} \qquad \text{(by(4.7))}$$

$$= \sum_{n=0}^{\infty} \Omega(n)q^n. \qquad \square$$

Corollary. For $n > 0$,

(4.10) $$p_n(gn,\mathcal{P}|q) = \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix} q^j.$$

Remark. Equation (4.10) identifies the RS-polynomial in this instance as a special case of the Rogers-Szegö polynomials (Andrews; 1976, p. 49).

Proof. By (4.7) and (2.2)

(4.11) $$\sum_{n=0}^{\infty} p_n(gn,\mathcal{P}|q)t^n = M\Theta_3(q,t)$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{(tq)_m}$$

$$= 1 + \sum_{m=1}^{\infty} t^m \sum_{j=0}^{\infty} \begin{bmatrix} m+j-1 \\ j \end{bmatrix} t^j q^j \qquad \text{(by (3.7))}$$

$$= 1 + \sum_{n=1}^{\infty} t^n \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix} q^j .$$

Comparing coefficients of t^n in the extremes of (4.11), we obtain (4.10). □

Most of the other third order mock theta functions can be treated in a similar manner. However for the most part the combinatorial interpretations of them are straight forward and are covered well in N. J. Fine's soon to be published book (Fine; 1985).

5. The Fifth and Seventh Order Mock Theta Functions.

We lump together the fifth order mock theta functions with the seventh order functions. These families first arose in Ramanujan's last letter to Hardy (see Ramanujan; 1927, pp. 354-355). G. N. Watson (1937) proved most of the assertions about the fifth order mock theta functions. There are alternative combinatorial interpretations available for the fifth order mock theta functions. We shall concentrate on the one which serves as a prototype for the seventh order mock theta functions and other applications. To this end we define an arithmetic function $\rho_{\lambda;a,b}$ on all partitions π by

$$\text{(5.1)} \qquad \rho_{\lambda;a,b}(\pi) = \max(\lambda \cdot \ell(\pi) - a, \lambda \cdot \#(\pi) - b),$$

where $\ell(\pi)$ is the largest part of π and $\#(\pi)$ is the number of parts of π.

The two fifth order mock theta functions of most interest to us here are (G. N. Watson; 1937, p. 277)

$$\text{(5.2)} \qquad f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}$$