London Mathematical Society Lecture Note Series. 105

A Local Spectral Theory for Closed Operators

IVAN ERDELYI

Temple University

WANG SHENGWANG

Nanjing University

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York New Rochelle

Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
10, Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1985

Library of Congress catalogue card number: 85-47812

British Library cataloguing in publication data

Erdelyi, I.
A local spectral theory for closed operators- (London Mathematical Society lecture note series, ISSN 0076-0052; 105)
1. Banach spaces 2. Spectral theory (mathematics)
I. Title II. Wang, Shengwang III. Series
515.7'32 QA322.2

ISBN 0 521 31314 7

Transferred to digital printing 2001

CONTENTS

PREFACE

Glossary of Notations and Symbols

Chapter I. INTRODUCTION 1

§. 1. The spectral decomposition problem 1

§. 2. The single valued extension property 5

§. 3. Invariant subspaces. General properties 10

§. 4. Invariant subspaces. Special properties 16

Notes and Comments 31

Chapter II. THE SPECTRAL DECOMPOSITION PROPERTY 33

§. 5. The 1-spectral decomposition property 33

§. 6. The equivalence of the 1-SDP and the SDP 50

§. 7. Spectral elements in functional calculus 53

Notes and Comments 57

Chapter III. SPECTRAL DUALITY 61

§. 8. The duality theorem 61

§. 9. The predual theorem 63

§.10. The analytically invariant subspace in duality 73

Notes and Comments 79

Chapter IV. SPECTRAL RESOLVENTS 81

§.11. Spectral resolvents. General properties 81

§.12. Monotonic spectral resolvents 87

§.13. Analytically invariant spectral resolvents 92

§.14. Spectral capacities 101

Notes and Comments 105

Chapter V. SPECIAL TOPICS IN SPECTRAL DECOMPOSITION 108

§.15. The strong spectral decomposition property 108

§.16. Strong spectral resolvents 123

§.17. Spectral decompositions with respect to the identity 129

Notes and Comments 147

Appendix A. The (**)-version of the predual theorem 148

Appendix B. Some open problems 161

BIBLIOGRAPHY 162

Index 177

CHAPTER I. INTRODUCTION

§.1. THE SPECTRAL DECOMPOSITION PROBLEM.

In extending the spectral theory beyond the class of normal operators, N. Dunford [Du.1958] gives a formal definition of the *spectral decomposition* (reduction) problem, for a closed operator T acting on a complex Banach space X. To interpret Dunford's definition, one has to express X as a finite direct sum of invariant subspaces X_i, such that the spectra of the restrictions $T|X_i$ be contained in some given closed sets.

For the class of *spectral operators*, the spectral decomposition is accomplished with the help of a *spectral measure* E: a homomorphic map of the σ-algebra of the Borel sets of the complex plane $\mathbb{C}$ into the Boolean algebra of the projection operators on X, with unit $I = E(\mathbb{C})$. A spectral measure, countably additive in the strong operator topology, is uniquely determined by T (for unbounded spectral operators, see e.g. [Ba.1954]), and is referred to as the *resolution of the identity* (or spectral resolution) of T.

A generalization of the spectral operator concept is due to C. Foiaş [Fo.1960], who replaced the role of the spectral measure by that of the *spectral distribution* (for a full dress account of this theory see [C.1968] and [C-Fo.1968]).

By making the spectral theory independent of such external constraints as direct sum decomposition, spectral measure and spectral distribution, we adopt the following

1.1. DEFINITION. Given a closed operator $T : \mathcal{D}_T(\subset X) \to X$, a *spectral decomposition* of X by T is a finite system

$$\{(G_i, X_i)\} \subset \mathcal{G} \times \text{Inv } T$$

satisfying the following conditions:

(1) $\{G_i\} \in \text{cov } \sigma(T)$;

(2) $X_i \subset \mathcal{D}_T$, if G_i is relatively compact ($G_i \in \mathcal{G}^K$);

(3) $X = \sum_i X_i$;

(4) $\sigma(T|X_i) \subset \overline{G}_i$ or, equivalently,

$\sigma(T|X_i) \subset G_i$ for all i.

Special properties of the invariant subspaces X_i induce special types of spectral decompositions. At center stage in this circle of ideas is the concept of *spectral decomposition property*. In the spirit of 1.1, we see that a spectral decomposition of X by T is formally operated by a map $E : G \to \text{Inv } T$, with the original family G providing the sets $\overline{G}_i$ to contain the spectra of the restrictions $T|E(G_i)$ and the final family Inv T supplying the summands X_i for the linear sum decomposition of X. While we defer the definition and the study of the map E until we reach Chapter IV, we indulge in a little discussion of our program.

In this work, we are primarily interested in extending the general spectral decomposition problem to the case of unbounded closed operators. When working with unbounded operators, the point at infinity of the one-point compactified complex plane $\mathbb{C}_\infty$ assumes a special role. If T is bounded then the resolvent operator is analytic at ∞ and

$$R(\infty;T) = \lim_{\lambda\to\infty} R(\lambda;T) = 0.$$

For unbounded T, $R(\cdot;T)$ has a singularity at ∞ and, for some purpose, the extended spectrum $\sigma_\infty(T) = \sigma(T) \cup \{\infty\}$ may be conveniently used. In many cases, however, when working within the topology of $\mathbb{C}$, we may still employ the ordinary spectrum $\sigma(T)$ of an unbounded T.

The basic requirement imposed by any spectral theory is the existence of proper invariant subspaces. A proper invariant subspace Y under T may be a summand of the spectral decomposition. Also Y produces the restriction $T|Y$ and the coinduced operator T/Y on the quotient space X/Y. Certain properties of $T|Y$ and T/Y may characterize the invariant subspace Y. The general and some special properties of invariant subspaces will be examined in terms of restriction and coinduced operators.

Some properties of bounded operators can be carried over to the unbounded case. Thus, in a few instances, the bounded case techniques can be adapted to unbounded operators. Most of the times, however, the proofs concerning unbounded operators are intrinsically different.

Originally, the concept of *decomposable operator* has been defined and its theory developed for bounded operators on a Banach space ([Fo.1963] and a wide variety of other papers). An extension of this concept to the unbounded case [V.1969; 1971,a] as well as to operators on a Fréchet space [V.1971] has been achieved by separating the part of the spectrum on which T failed to have the *single valued extension property*.

The minimal closed subset S of $\sigma_\infty(T)$ on whose complement T has the single valued extension property, in the sense that for any analytic function $f : \omega(\subset S^c) \to \mathcal{D}_T$, $(\lambda-T)f(\lambda) = 0$ on $\omega \cap \mathbb{C}$ implies $f(\lambda) = 0$ on ω, is called the *spectral residuum* of T. An open S-*cover* of $\sigma_\infty(T)$ has all but one member G_S disjoint from S. The pathology lurking in an S-cover is brought out in the pertinent spectral decomposition, by the invariant subspace that corresponds to G_S. The theory of these S- (or *residually*-) *decomposable operators* gives an interesting insight into the structure of some non-decomposable operators (e.g. [B.1975], [N.1979; 1979-1980; 1980], [T.1983], [V.1969; 1971; 1971,a; 1982], [W.1984], [W-Li.1984]).

To reach our targets, we shall follow a different path. The key ingredient in our approach is the spectral decomposition property [E-L.1978]. This property endows a closed operator with the single valued extension property [E.1980,a] and in the light of some subsequent works as [A.1979], [L.1981], [N.1978] and [Sh.1979], it gives a simpler and more natural interpretation to the concept of bounded decomposable operator. How much remains true if we drop the assumption of boundedness? The answer will arise from the spectral manifold discernibly at work with bounded and unbounded operators. What we loose is the property of the spectral manifold to be $\{0\}$ at the empty set and this fact distinguishes between the operators with the spectral decomposition property which are decomposable and which are not.

The substance of the spectral theory suggests that a certain duality exists between an operator and its conjugate. It is a major topic of this work to explore the spectral duality of unbounded closed operators.

The use of certain spectral constructs, such as (*pre-*) *spectral capacity* and (*pre-*) *spectral resolvent* simplifies certain proofs and gives new characterizations to operators which possess some specific spectral properties. The interplay between various spectral resolvents and contingent properties of the operators will answer some questions and will open new problems.

For a deeper analysis, we shall frequently use the *contour integral* of a vector-valued function. An open $\Delta \subset \mathbb{C}$ is a *Cauchy domain* if it has a finite number of components and its boundary $\Gamma = \partial\Delta$ is a positively oriented finite system of closed, mutually nonintersecting rectifiable Jordan arcs. Γ will be referred to as an *admissible contour*. If a Cauchy domain Δ is a neighborhood of a set $S \subset \mathbb{C}$, we shall simply say that Δ is a *Cauchy neighborhood* of S. A set $H \subset \mathbb{C}$ is referred to as a *neighborhood of* ∞, in symbols $H \in V_\infty$, if the closure of its complement H^c is compact in $\mathbb{C}$. An open set ω is a neighborhood of $S \cup \{\infty\}$ if it is a neighborhood of both S and ∞. Without loss of generality, we assume that for $S \subset \mathbb{C}$, any $\{G_i\}_{i=0}^n \in \text{cov}\, S$ has, at most, one unbounded set G_0.

We denote by A_T the class of functions $f : \omega_f(\subset \mathbb{C}) \to \mathbb{C}$ which are locally analytic on a neighborhood ω_f of $\sigma_\infty(T)$ and regular at ∞. For $f \in A_T$, we write $f(\infty) = \lim_{\lambda\to\infty} f(\lambda)$. For the use of the contour integral it will be *implicitly assumed* that $\rho(T) \neq \emptyset$.

The functional calculus is established by the algebraic homomorphism $f \to f(T)$ between the algebra A_T and the *Banach algebra* $B(X)$ of bounded linear operators which map X into X. If $f \in A_T$, then

$$f(T) = f(\infty) + \frac{1}{2\pi i} \int_{\partial\Delta} f(\lambda)R(\lambda;T)d\lambda,$$

where Δ is any Cauchy neighborhood of $\sigma_\infty(T)$ with $\bar{\Delta} \subset \omega_f$. If $f(\infty) = 0$, the range of $f(T)$ is contained in $\mathcal{D}_T$.

It is an open question whether every bounded and unbounded closed operator on a Banach space has a proper invariant subspace. In fact, in terms of reducing subspaces, the unilateral shift operator on ℓ^2 is a counterexample [H.1951], [H.1967]. The existence of proper invariant subspaces for *subnormal operators* was proved by S. Brown [Bro.1979], (see also [St.1979]) and some extensions were obtained in [St.1980], [Ap.1980] and [Ap.1981]. In defining the spectral decomposition property of the given closed operator T, it will be assumed the existence of the necessary number of invariant subspaces with the required spectral property (as mentioned in the first paragraph of this section). The fact that we are not working in void will become soon evident by the wealth of the spectral maximal and T-bounded spectral maximal spaces, as constructive elements of the spectral decomposition problem.

§.2. THE SINGLE VALUED EXTENSION PROPERTY

The spectral decomposition problem cannot be properly studied without the single valued extension property (SVEP). This property has a profound effect on both the operators and the invariant subspaces involved in a spectral decomposition.

2.1. DEFINITION. A linear operator $T : \mathcal{D}_T(\subset X) \to X$ is said to have the SVEP if, for every analytic† function $f : \omega_f \to \mathcal{D}_T$ defined on an open $\omega_f \subset \mathbb{C}$, the condition $(\lambda-T)f(\lambda) \equiv 0$ implies $f(\lambda) \equiv 0$.

Equivalently, for each $x \in X$, any two analytic functions $f : \omega_f \to \mathcal{D}_T$, $g : \omega_g \to \mathcal{D}_T$, satisfying condition

$$(\lambda-T)f(\lambda) = (\lambda-T)g(\lambda) = x \quad \text{on} \quad \omega_f \cap \omega_g,$$

agree on $\omega_f \cap \omega_g$. When this property holds, the union of the domains ω_f of all $\mathcal{D}_T$-valued analytic functions f, which identically verify equation

$$(2.1) \qquad (\lambda-T)f(\lambda) = x,$$

is called the *local resolvent set* and is denoted by $\rho(x,T)$. The SVEP implies the existence of a (maximally extended) analytic function $x_T(\cdot)$, or $x(\cdot)$ if T is understood, referred to as *local resolvent*, which maps $\rho(x,T)$ into $\mathcal{D}_T$ and identically verifies (2.1). The *local spectrum* $\sigma(x,T)$, defined as the complement in $\mathbb{C}$ of $\rho(x,T)$, is the set of singularities of $x(\cdot)$.

If not specified otherwise, we shall henceforth assume that the given T is a *closed operator*.

2.2. THEOREM. Given T, for every $x \in X$ and $\lambda_0 \in \mathbb{C}$, the following assertions are equivalent:

(I) there is a neighborhood δ of λ_0 and an analytic function $f : \delta \to \mathcal{D}_T$ verifying equation (2.1) on δ;

(II) there are numbers $M > 0$, $R > 0$ and a sequence $\{a_n\}_{n=0}^{\infty} \subset \mathcal{D}_T$ with the following properties:

$$(2.2) \quad (a) \ (\lambda_0-T)a_0 = x; \quad (b) \ (\lambda_0-T)a_{n+1} = a_n; \quad (c) \ \|a_n\| \le MR^n, \ n \in \mathbb{Z}^+.$$

PROOF. (I) => (II): We may assume that

$$\delta = \{\lambda : |\lambda-\lambda_0| < r\} \quad \text{for some} \quad r > 0.$$

† the term "analytic" will be indistinguishably used for "locally analytic".

Let

$$f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0-\lambda)^n, \quad \lambda \in \delta \tag{2.3}$$

be the power series expansion of f. By decreasing r, we may assume that (2.3) holds on $\overline{\delta}$. Then, for the radius r of $\partial\delta$, $\|a_n r^n\| \to 0$ as $n \to \infty$. Hence there is $M > 0$ such that

$$\|a_n\| r^n \leq M, \quad n \in Z^+. \tag{2.4}$$

For $R = r^{-1}$, (2.4) implies (2.2,c). By making $\lambda = \lambda_0$ in (2.1) and (2.3), one obtains (2.2,a). Furthermore, it follows from (2.3) that

$$a_n = -\frac{1}{2\pi i}\int_{\partial\delta} \frac{f(\lambda)}{(\lambda_0-\lambda)^{n+1}}\, d\lambda, \quad n \in Z^+.$$

In view of (2.1), one can write

$$(\lambda_0-T)f(\lambda) = (\lambda_0-\lambda)f(\lambda) + (\lambda-T)f(\lambda) = (\lambda_0-\lambda)f(\lambda) + x.$$

Since T is closed, one obtains $a_n \in \mathcal{D}_T$ ($n \in Z^+$) and

$$(\lambda_0-T)a_{n+1} = -\frac{1}{2\pi i}\int_{\partial\delta} \frac{(\lambda_0-T)f(\lambda)}{(\lambda_0-\lambda)^{n+2}}\, d\lambda$$

$$= -\frac{1}{2\pi i}\int_{\partial\delta} \frac{f(\lambda)}{(\lambda_0-\lambda)^{n+1}}\, d\lambda - \frac{1}{2\pi i}\int_{\partial\delta} \frac{x}{(\lambda_0-\lambda)^{n+2}}\, d\lambda = a_n.$$

This proves (2.2,b).

(II) => (I): In view of (2.2,c), the series (2.3) defines a function f, analytic on

$$\delta = \{\lambda : |\lambda-\lambda_0| < R^{-1}\}.$$

Thus, for

$$f_k(\lambda) = \sum_{n=0}^{k} a_n(\lambda_0-\lambda)^n, \quad \lambda \in \delta, \ k \in \mathbf{N},$$

with the help of (2.2,a), one obtains

$$(\lambda-T)f_k(\lambda) = \sum_{n=0}^{k} (\lambda-T)a_n(\lambda_0-\lambda)^n = \sum_{n=0}^{k} (\lambda_0-T)a_n(\lambda_0-\lambda)^n$$

$$- \sum_{n=0}^{k} a_n(\lambda_0-\lambda)^{n+1} = x + \sum_{n=1}^{k} a_{n-1}(\lambda_0-\lambda)^n - \sum_{n=0}^{k} a_n(\lambda_0-\lambda)^{n+1}$$

$$= x - a_k(\lambda_0-\lambda)^{k+1}.$$

For every $\lambda \in \delta$, $f_k(\lambda) \to f(\lambda)$ and $a_k(\lambda_0-\lambda)^{k+1} \to 0$, as $k \to \infty$. Since T is closed, $f(\lambda) \in \mathcal{D}_T$ and $(\lambda-T)f(\lambda) = x$, for all $\lambda \in \delta$. □

2.3. COROLLARY. T does not have the SVEP iff there exists $\lambda_0 \in \mathbb{C}$ and there are numbers $M > 0$, $R > 0$ and a sequence $\{a_n\}_{n=0}^{\infty} \subset \mathcal{D}_T$ such that

(2.5) $(\lambda_0-T)a_0 = 0$; $(\lambda_0-T)a_{n+1} = a_n$; $\|a_n\| \leq MR^n$ $(n \in Z^+)$; $a_n \neq 0$ for some n.

PROOF. T does not have the SVEP iff, for some analytic function $f : \omega_f \to \mathcal{D}_T$ and $\lambda_0 \in \omega_f$, there is a neighborhood δ of λ_0 such that

(2.6) $$(\lambda-T)f(\lambda) \equiv 0 \quad \text{and} \quad f(\lambda) \not\equiv 0 \quad \text{on } \delta.$$

In view of 2.2, the situation described by (2.6) occurs iff conditions (2.5) hold. □

2.4. COROLLARY. T does not have the SVEP if there is $\lambda_0 \in \mathbb{C}$ such that λ_0-T is surjective but not injective.

PROOF. Assume that λ_0-T is surjective but not injective. Since λ_0-T is closed, it follows from the open mapping theorem that there is $R > 0$ such that, for each $y \in X$, there is $x \in \mathcal{D}_T$ satisfying conditions

$$(\lambda_0-T)x = y, \quad \|x\| \leq R\|y\|.$$

Since λ_0-T is not injective, one can choose $a_0 \in \mathcal{D}_T$ with $\|a_0\| = 1$ and $(\lambda_0-T)a_0 = 0$. For each $n \in \mathbb{N}$, let $a_n \in \mathcal{D}_T$ satisfy conditions

$$(\lambda_0-T)a_n = a_{n-1}, \quad \|a_n\| \leq R\|a_{n-1}\|.$$

Then (2.5) holds for M = 1 and hence T does not have the SVEP. □

2.5. COROLLARY. Let T have the SVEP. Then $\lambda \in \sigma(T)$ iff λ-T is not surjective.

2.6. PROPOSITION. If T has the SVEP then the following properties hold:

(I) $\sigma(x+y,T) \subset \sigma(x,T) \cup \sigma(y,T)$; $x,y \in X$;

(II) $ax(\lambda) + by(\lambda) = (ax+by)(\lambda)$; $a,b \in \mathbb{C}$; $x,y \in X$; $\lambda \in \rho(x,T) \cap \rho(y,T)$;

(III) $\sigma(Ax,T) \subset \sigma(x,T)$ for every $A \in B(X)$ which commutes with T;

(IV) $\sigma(Tx,T) \subset \sigma(x,T)$ and $(Tx)(\lambda) = Tx(\lambda)$, $x \in \mathcal{D}_T$, $\lambda \in \rho(x,T)$;

(V) $\sigma[x(\lambda),T] = \sigma(x,T)$; $x \in X$, $\lambda \in \rho(x,T)$;

(VI) for any $A \in B(X)$ with the SVEP, $\sigma(x,A) = \emptyset$ iff $x = 0$.

PROOF. Properties (I), (II), (III) and (VI) can be proved as in the bounded case (e.g. [Du-S.1971; 1, XVI. 2.1, 2.2]).

(IV): It follows from

$$(\lambda-T)x(\lambda) = x, \quad x \in \mathcal{D}_T, \lambda \in \rho(x,T)$$

that $Tx(\lambda) \in \mathcal{D}_T$ and that $Tx(\cdot)$ is analytic on $\rho(x,T)$. Then

$$(\lambda-T)Tx(\lambda) = T(\lambda-T)x(\lambda) = Tx$$

implies (IV).

(V): Given $x \in X$, for every $\lambda \in \rho(x,T)$, there is an analytic function $\xi_\lambda : \rho[x(\lambda),T] \to \mathcal{D}_T$ verifying equation

$$(\mu-T)\xi_\lambda(\mu) = x(\lambda) \quad \text{on} \quad \rho[x(\lambda),T]. \tag{2.7}$$

Since, for $\lambda \in \rho(x,T)$, $x(\lambda) \in \mathcal{D}_T$, (2.7) implies that $T\xi_\lambda(\mu) \in \mathcal{D}_T$ and

$$(\mu-T)(\lambda-T)\xi_\lambda(\mu) = (\lambda-T)(\mu-T)\xi_\lambda(\mu) = (\lambda-T)x(\lambda) = x.$$

Since $(\lambda-T)\xi_\lambda(\mu) = (\lambda-\mu)\xi_\lambda(\mu) + x(\lambda)$ is analytic on $\rho[x(\lambda),T]$, we have $\mu \in \rho(x,T)$. Thus $\sigma(x,T) \subset \sigma[x(\lambda),T]$.

Conversely, for $\lambda \in \rho(x,T)$, define the analytic function $g_\lambda : \rho(x,T) \to X$, by

$$g_\lambda(\mu) = \begin{cases} -\dfrac{x(\mu) - x(\lambda)}{\mu-\lambda}, & \text{if } \mu \neq \lambda; \\ -x'(\lambda), & \text{if } \mu = \lambda. \end{cases}$$

For $\mu \neq \lambda$, we have $g_\lambda(\mu) \in \mathcal{D}_T$ and

$$(\mu-T)g_\lambda(\mu) = -\frac{x}{\mu-\lambda} + x(\lambda) + \frac{x}{\mu-\lambda} = x(\lambda). \tag{2.8}$$

T being closed, by letting $\mu \to \lambda$, one has $x'(\lambda) \in \mathcal{D}_T$ and hence (2.8) also holds for $\mu = \lambda$. Consequently, $\sigma[x(\lambda),T] \subset \sigma(x,T)$ and (V) follows. □

2.7. PROPOSITION. If T has the SVEP, then

$$\sigma(T) = \bigcup\{\sigma(x,T) : x \in X\}.$$

PROOF. Let $\lambda \in \mathbb{C} - \bigcup\{\sigma(x,T) : x \in X\}$. For every $x \in X$, $(\lambda-T)x(\lambda) = x$ implies that $\lambda-T$ is surjective. Then $\lambda \in \rho(T)$, by 2.5. Consequently, $\sigma(T) \subset \bigcup\{\sigma(x,T) : x \in X\}$. The opposite inclusion is obvious. □

If T has the SVEP, then 2.6 implies that, for every $H \subset \mathbb{C}$,

$$X(T,H) = \{x \in X : \sigma(x,T) \subset H\} \tag{2.9}$$

is a linear manifold in X. Moreover, if X(T,H) is closed then, by 2.6 (III) and (IV), X(T,H) is a *hyperinvariant* subspace under T (i.e. X(T,H) is invariant under every $A \in B(X)$ which commutes with T). The linear manifold (2.9) is called a *spectral manifold* (of T).

The SVEP is stable under functional calculus and the proof for an unbounded T follows by lines similar to that for a bounded operator [C-Fo.1967,1968].

2.8. PROPOSITION. Let T be such that $\rho(T) \neq \emptyset$. If T has the SVEP then, for each $f \in A_T$, f(T) has the same property. Conversely, if $f \in A_T$ is nonconstant on every component of its domain and if f(T) has the SVEP, then T has the SVEP.

The localized version of the spectral mapping theorem [Ap.1968] or [Bar-Ka.1973] has its extension to unbounded operators as given in [V.1982, IV. Theorem 3.12]. See also [Ho.1983,a].

2.9. PROPOSITION. Given T, let $f \in A_T$ be nonconstant on every component of its domain. Then, for every $x \in X$,

$$\sigma[x,f(T)] = f[\sigma_\infty(x,T)].$$

2.10. COROLLARY. Given T, let $f \in A_T$ be nonconstant on every component of its domain. If T has the SVEP then, for every set $H \subset \mathbb{C}$,

$$X[f(T),H] \subset X[T,f^{-1}(H)] \subset X[f(T),H \cup f(\{\infty\})]. \tag{2.10}$$

In particular, if $f(\infty) \in H$, then

$$X[f(T),H] = X[T,f^{-1}(H)]. \tag{2.11}$$

PROOF. By 2.9, for every $x \in X[f(T),H]$, we have

$$f[\sigma(x,T)] \subset \sigma[x,f(T)] \subset H$$

and hence $\sigma(x,T) \subset f^{-1}(H)$. Thus $x \in X[T,f^{-1}(H)]$ and the first inclusion of (2.10) follows. Next, let $x \in X[T,f^{-1}(H)]$. Then $\sigma(x,T) \subset f^{-1}(H)$ and (2.9) implies

$$\sigma[x,f(T)] = f[\sigma_\infty(x,T)] \subset f[f^{-1}(H) \cup \{\infty\}] \subset H \cup f(\{\infty\}).$$

Hence, the second inclusion of (2.10) is obtained. Now (2.11) is a direct consequence of $f(\infty) \in H$ and (2.10). □

§.3. INVARIANT SUBSPACES. GENERAL PROPERTIES.

To develop the constructive elements of the spectral decomposition of X, we devote this section to a spectral-theoretic study of invariant subspaces. A subspace Y of X is *invariant* under T, in symbols, $Y \in \text{Inv } T$, if $T(Y \cap \mathcal{D}_T) \subset Y$. An invariant subspace Y produces two operators: the restriction $T|Y$ and the coinduced $\hat{T} = T/Y$ by T on the quotient space X/Y. The latter has the domain

$$\mathcal{D}_{\hat{T}} = \{\hat{x} \in X/Y : \hat{x} \cap \mathcal{D}_T \neq \emptyset\}$$

and, for $\hat{x} \in \mathcal{D}_{\hat{T}}$, $x \in \hat{x} \cap \mathcal{D}_T$, we define $\hat{T}\hat{x} = (Tx)^\wedge$.

3.1. PROPOSITION. Given T and $Y \in \text{Inv } T$, consider the following conditions:

$$\sigma(T) \cup \sigma(T|Y) \neq \mathbb{C}; \tag{3.1}$$

$$\hat{T} = T/Y \text{ is closed on } X/Y. \tag{3.2}$$

Then (3.1) implies (3.2) and either of them produces the following inclusions

$$\begin{aligned} &\sigma(\hat{T}) \subset \sigma(T) \cup \sigma(T|Y); \\ &\sigma(T|Y) \subset \sigma(T) \cup \sigma(\hat{T}); \\ &\sigma(T) \subset \sigma(\hat{T}) \cup \sigma(T|Y). \end{aligned} \tag{3.3}$$

PROOF. Assume (3.1) and let $\lambda \in \rho(T) \cap \rho(T|Y)$ be arbitrary. For any $x \in Y$, we have $R(\lambda;T)x = R(\lambda;T|Y)x \in Y$ and hence Y is invariant under $R_\lambda = R(\lambda;T)$. Let $\hat{R}_\lambda$ be the coinduced operator by R_λ on X/Y. The identities

$$(\lambda-T)R_\lambda x = x,\ x \in X;\quad R_\lambda(\lambda-T)x = x,\ x \in \mathcal{D}_T$$

give rise to

$$(\lambda-\hat{T})\hat{R}_\lambda\hat{x} = \hat{x},\ \hat{x} \in X/Y;\quad \hat{R}_\lambda(\lambda-\hat{T})\hat{x} = \hat{x},\ \hat{x} \in \mathcal{D}_{\hat{T}}. \tag{3.4}$$

It follows from (3.4) that $\hat{R}_\lambda$ is the inverse of $\lambda-\hat{T}$. Since $\hat{R}_\lambda$ is bounded and defined on X/Y, it is closed and hence $\hat{T}$ is closed. Furthermore, by

(3.4), $\lambda \in \rho(\hat{T})$ and this implies (3.3). The remainder of the proof is routine and we omit it. □

3.2. PROPOSITION. Given T, let $X_0, X_1, Y \in \text{Inv } T$ satisfy the following conditions

$$X = X_0 + X_1, \quad X_1 \subset \mathcal{D}_T \cap Y; \tag{3.5}$$

$$\sigma(T|X_0) \subset F, \quad \sigma(T|X_0 \cap Y) \subset F, \tag{3.6}$$

for some closed $F \subsetneq \mathbb{C}$. Then $\hat{T} = T/Y$ is closed. Moreover, if $\tilde{T} = (T|X_0)/Y \cap X_0$ (i.e. $\tilde{T}$ is the coinduced operator by $T|X_0$ on $X_0/Y \cap X_0$), then $\hat{T}$ and $\tilde{T}$ are similar and hence

$$\sigma(\hat{T}) = \sigma(\tilde{T}). \tag{3.7}$$

PROOF. Since, by (3.6),

$$\sigma(T|X_0) \cup \sigma(T|Y \cap X_0) \subset F \neq \mathbb{C},$$

$\tilde{T}$ is closed, by 3.1. Next, we show that $\hat{T}$ and $\tilde{T}$ are similar. In view of (3.5), each $x \in \mathcal{D}_T$ has a representation

$$x = x_0 + x_1 \quad \text{with} \quad x_i \in X_i, \quad i=0,1.$$

Since $x_1 \in \mathcal{D}_T$, we have $x_0 \in \mathcal{D}_T$. Therefore, $x_1 \in Y \cap \mathcal{D}_T$ and $x_0 \in X_0 \cap \mathcal{D}_T$. For $x \in X$, let $\hat{x} = x + Y \in X/Y$ and, for $x_0 \in X_0$, let $\tilde{x}_0 = x_0 + Y \cap X_0 \in X_0/Y \cap X_0$. The spaces X/Y, $X_0/Y \cap X_0$ are topologically isomorphic. Let $A : X/Y \to X_0/Y \cap X_0$, with $A\hat{x} = \tilde{x}_0$, be the topological isomorphism. For every $\hat{x} \in \mathcal{D}_{\hat{T}}$, there is $x \in \hat{x} \cap \mathcal{D}_T$ with $A\hat{x} = \tilde{x}_0 \in \mathcal{D}_{\tilde{T}}$. Conversely, for every $\tilde{x}_0 \in \mathcal{D}_{\tilde{T}}$, there is $x_0 \in \tilde{x}_0 \cap (X_0 \cap \mathcal{D}_T)$ and hence $\hat{x} \in \mathcal{D}_{\hat{T}}$. Consequently, $A\mathcal{D}_{\hat{T}} = \mathcal{D}_{\tilde{T}}$. For each $\hat{x} \in \mathcal{D}_{\hat{T}}$, one obtains

$$A\hat{T}\hat{x} = A(Tx)^{\wedge} = (Tx_0)^{\sim} = \tilde{T}\tilde{x}_0 = \tilde{T}A\hat{x}.$$

Hence $\hat{T}$ and $\tilde{T}$ are similar. Consequently, $\hat{T}$ is closed and (3.7) holds. □

3.3. COROLLARY. Given T, let $X_0, X_1 \in \text{Inv } T$ be such that

$$X = X_0 + X_1, \quad X_1 \subset \mathcal{D}_T; \quad \sigma(T|X_0) \subset F, \quad \sigma(T|X_0 \cap X_1) \subset F,$$

for some closed $F \subsetneq \mathbb{C}$. Then T/X_1 is closed and

$$\sigma(T/X_1) \subset \sigma(T|X_0) \cup \sigma(T|X_0 \cap X_1).$$

PROOF. For $Y = X_1$, the corollary is a direct consequence of 3.2. □

3.4. PROPOSITION. Given T, let $X_0, X_1 \in \operatorname{Inv} T$ be such that

$$X = X_0 + X_1, \quad X_1 \subset \mathcal{D}_T.$$

Then $\hat{T} = T/X_0$ is bounded and

$$\sigma(\hat{T}) \subset \sigma(T|X_1) \cup \sigma(T|X_0 \cap X_1). \tag{3.8}$$

PROOF. Let $x \in X$ and put $\hat{x} = x + X_0$. It follows from

$$x = x_0 + x_1, \quad x_0 \in X_0, \quad x_1 \in X_1 \subset \mathcal{D}_T,$$

that $x_1 \in \hat{x} \cap \mathcal{D}_T$. Therefore, $\hat{x} \cap \mathcal{D}_T \neq \emptyset$. Thus $\hat{x} \in \mathcal{D}_{\hat{T}}$ and hence $\mathcal{D}_{\hat{T}} = X/X_0$. Since the quotient spaces X/X_0 and $X_1/X_0 \cap X_1$ are topologically isomorphic, $\hat{T}$ and $\tilde{T} = (T|X_1)/X_0 \cap X_1$ are similar, by 3.2. Since, by 3.2, $\tilde{T}$ is closed, so is $\hat{T}$. Thus $\hat{T}$ is bounded, by the closed graph theorem. Since $\sigma(\hat{T}) = \sigma(\tilde{T})$, (3.8) follows from (3.3). □

We recall that if $T : \mathcal{D}_T(\subset X) \to Y$ and $A : \mathcal{D}_A(\subset Y^*) \to X^*$ are adjoint to each other and one of them is densely defined, then the other is closable (e.g. [Kat.1966, III.5. Theorem 5.28]). The following proposition gives a condition for a coinduced T/Y to be closable on X/Y.

3.5. PROPOSITION. Let T be densely defined and let $Y \in \operatorname{Inv} T$ be such that $\overline{Y \cap \mathcal{D}_T} = Y$. If $T^*|Y^a$ is densely defined then T/Y is closable. Moreover,

$$(T/Y)^* = T^*|Y^a, \quad (Y^a \text{ is the annihilator of } Y \text{ in } X^*).$$

PROOF. The fact that $Y^a \in \operatorname{Inv} T^*$, follows easily. Indeed, for $x \in Y \cap \mathcal{D}_T$ and $x^* \in Y^a \cap \mathcal{D}_{T^*}$, we have $0 = \langle Tx, x^* \rangle = \langle x, T^*x^* \rangle$. Since $\overline{Y \cap \mathcal{D}_T} = Y$, one obtains $T^*x^* \in Y^a$.

Y^a can be viewed as the dual of X/Y, under the isometric isomorphism $(X/Y)^* \to Y^a$. For convenience, we make no distinction between Y^a and $(X/Y)^*$, and denote by $\langle \hat{x}, x^* \rangle$ the linear functional $x^* \in Y^a$ on X/Y. For $\hat{x} \in \mathcal{D}_{T/Y}$, $x \in \hat{x} \cap \mathcal{D}_T$, $y \in Y$ and $x^* \in Y^a \cap \mathcal{D}_{T^*}$, one obtains

$$\begin{aligned} \langle (T/Y)\hat{x}, x^* \rangle &= \langle (Tx)^\wedge, x^* \rangle = \langle Tx + y, x^* \rangle = \langle Tx, x^* \rangle \\ &= \langle x, T^*x^* \rangle = \langle x + y, T^*x^* \rangle = \langle \hat{x}, T^*x^* \rangle. \end{aligned} \tag{3.9}$$

Consequently, T/Y and $T^*|Y^a$ are conjugate to each other and since $T^*|Y^a$ is densely defined, T/Y is closable.

To prove the second statement, note that $\overline{\mathcal{D}}_T = X$ implies that $\overline{\mathcal{D}}_{T/Y} = X/Y$. Thus, T/Y is densely defined and hence the conjugate $(T/Y)^*$ exists. If $\mathcal{G}(\cdot)$ denotes the graph of an operator and $V\mathcal{G}(\cdot)$ is the inverse graph (i.e. the mapping $\mathcal{V} : X \times X \to X \times X$ is defined by $\mathcal{V}(x,y) = (-y,x)$), then it follows from (3.9) that

$$\mathcal{V}\mathcal{G}(T^*|Y^a) \subset \mathcal{G}(T/Y)^a = \mathcal{V}\mathcal{G}[(T/Y)^*]$$

and hence $(T/Y)^* \supset T^*|Y^a$. Now, let $x^* \in \mathcal{D}_{(T/Y)^*}$. For $x \in \mathcal{D}_T$ and $y \in Y$,

$$(3.10) \qquad \langle (T/Y)\hat{x},x^*\rangle = \langle \hat{x},(T/Y)^*x^*\rangle = \langle x+y,(T/Y)^*x^*\rangle = \langle x,(T/Y)^*x^*\rangle.$$

Thus, for every $x^* \in \mathcal{D}_{(T/Y)^*}$, $\langle (T/Y)\hat{x},x^*\rangle$ is a bounded linear functional on $\mathcal{D}_T$ and hence $x^* \in \mathcal{D}_{T^*}$. Furthermore, $x^* \in \mathcal{D}_{(T/Y)^*} \subset Y^a$ and hence $x^* \in Y^a \cap \mathcal{D}_{T^*}$. We have

$$(3.11) \qquad \langle (T/Y)\hat{x},x^*\rangle = \langle Tx + y, x^*\rangle = \langle Tx,x^*\rangle = \langle x,T^*x^*\rangle.$$

It follows from (3.10) and (3.11) that $(T/Y)^* \subset T^*|Y^a$. □

3.6. PROPOSITION. Given T, let $Y \in \text{Inv } T$. Then
(I) for any component G of $\rho(T)$, either $G \subset \sigma(T|Y)$ or $G \subset \rho(T|Y)$;
(II) if $Y \subset \mathcal{D}_T$, then each unbounded component of $\rho(T)$ is contained in $\rho(T|Y)$.

PROOF. (I). Suppose that we simultaneously have

$$\sigma(T|Y) \cap G \neq \emptyset \quad \text{and} \quad \rho(T|Y) \cap G \neq \emptyset.$$

Then $\partial\sigma(T|Y) \cap G \neq \emptyset$ and hence there is $\lambda \in \mathbb{C}$ such that

$$\lambda \in \partial\sigma(T|Y) \cap G \subset \sigma_a(T|Y) \subset \sigma_a(T) \subset \sigma(T).$$

This, however, is a contradiction.

(II). Since $Y \subset \mathcal{D}_T$, $\sigma(T|Y)$ is compact. If G is an unbounded component of $\rho(T)$ such that $\sigma(T|Y) \cap G \neq \emptyset$, then $\partial\sigma(T|Y) \cap G \neq \emptyset$. Now, an argument similar to that used in the proof of (I), leads one to a contradiction. □

We proceed with some elementary but useful properties of a closed operator acting on a direct sum decomposition of the underlying Banach space.

3.7. LEMMA. Let X be the direct sum of two subspaces X_1, X_2,

$$X = X_1 \oplus X_2$$

and let $T_i : \mathcal{D}_{T_i}(\subset X_i) \to X_i$ $(i=1,2)$ be closed operators. The linear operator $T : \mathcal{D}_T(\subset X) \to X$, defined by

$$\mathcal{D}_T = \{x \in X : x = x_1 + x_2,\ x_i \in \mathcal{D}_{T_i},\ i=1,2\},$$

$$Tx = T_1x_1 + T_2x_2, \quad x \in \mathcal{D}_T,$$

is closed and

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2).$$

PROOF. The proof is routine and is omitted. □

The *reducibility* of an operator in terms of *spectral sets* is now appropriate to be recorded. The theory is known (e.g. [Ta-La.1980,V.9.]).

3.8. THEOREM. Given T, if for closed disjoint sets σ_0, σ_1 with σ_0 compact, one has

$$\sigma(T) = \sigma_0 \cup \sigma_1$$

then there exist $X_0, X_1 \in \text{Inv } T$ satisfying the following conditions:

(I) $X = X_0 \oplus X_1$;

(II) $X_1 \subset \mathcal{D}_T$, $\sigma(T|X_i) = \sigma_i$ $(i=0,1)$.

3.9. COROLLARY. Given T, let $Y \in \text{Inv } T$ be such that $\sigma(T|Y)$ is compact. Then, there exist $\mathcal{T}, W \in \text{Inv } T$ with the following properties:

(i) $Y = \mathcal{T} \oplus W$;

(ii) $\mathcal{T} \subset \mathcal{D}_T$, $\sigma(T|\mathcal{T}) = \sigma(T|Y)$, $\sigma(T|W) = \emptyset$.

PROOF. For $\sigma_1 = \sigma(T|\mathcal{T})$, $\sigma_0 = \emptyset$, 3.8 applied to $T|Y$ gives rise to (i) and (ii). If Δ is a bounded Cauchy neighborhood of $\sigma(T|Y)$, the projection

$$Q = \frac{1}{2\pi i} \int_{\partial\Delta} R(\lambda;T|Y)d\lambda \tag{3.12}$$

produces $\mathcal{T} = QY$ and $W = (I_Y - Q)Y$, where I_Y is the identity in Y. □

3.10. LEMMA. Let X_1 and X_2 be subspaces of X such that

$$X = X_1 + X_2. \tag{3.13}$$

There is a constant $M > 0$ such that, for every $x \in X$ there is a representation

$$x = x_1 + x_2, \quad x_i \in X_i \ (i=1,2) \tag{3.14}$$

satisfying condition

$$\|x_1\| + \|x_2\| \le M \|x\|. \tag{3.15}$$

PROOF. Define the continuous map $P : X_1 \oplus X_2 \to X$, by

$$P(x_1 \oplus x_2) = x_1 + x_2,$$

equipped with the norm

$$\|x_1 \oplus x_2\| = \|x_1\| + \|x_2\|.$$

P is surjective, by (3.13). By the open mapping theorem, there is a constant $M > 0$ such that, for every $x \in X$ with (3.14), there exists

$$y = x_1 \oplus x_2 \in X_1 \oplus X_2$$

satisfying conditions

$$Py = x \quad \text{and} \quad \|y\| \leq M\|x\|.$$

Since $\|y\| = \|x_1\| + \|x_2\|$, (3.15) is obtained. □

3.11. PROPOSITION. Let X_1, X_2 be subspaces of X satisfying (3.13). If $f : \omega_f \to X$ is analytic on an open $\omega_f \subset \mathbb{C}$ then, for every $\lambda_0 \in \omega_f$, there is a neighborhood $\omega_0 (\subset \omega_f)$ of λ_0 and there are analytic functions $f_i : \omega_0 \to X_i$ $(i=1,2)$ such that

(3.16) $$f(\lambda) = f_1(\lambda) + f_2(\lambda) \quad \text{on } \omega_0.$$

PROOF. Put

$$\omega_0(\subset \omega_f) = \{\lambda : |\lambda - \lambda_0| < r\} \text{ for some } r > 0$$

and let

(3.17) $$f(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n, \quad \{a_n\} \subset X,$$

be the power series expansion of f in ω_0. By decreasing r, we may assume that

(3.18) $$\sup_n r^n \|a_n\| < \infty.$$

By 3.10, for every $n \in Z^+$, there is a representation

(3.19) $$a_n = a_{1n} + a_{2n}, \quad a_{in} \in X_i \quad (i=1,2)$$

with

(3.20) $$\|a_{1n}\| + \|a_{2n}\| \leq M\|a_n\| \text{ for some constant } M > 0.$$

In view of (3.18) and (3.20), the series

(3.21) $$f_i(\lambda) = \sum_{n=0}^{\infty} a_{in} (\lambda - \lambda_0)^n, \quad i=1,2$$

converge and hence the functions f_i are analytic on ω_0. Consequently, (3.17), (3.19) and (3.21) give rise to (3.16). □

§.4. INVARIANT SUBSPACES. SPECIAL PROPERTIES.

The time has come to have a closer look at the invariant subspaces which occur most frequently in spectral decompositions.

4.1. DEFINITION. Given T, $Y \in \text{Inv } T$ is said to be a *ν-space* of T if

$$\sigma(T|Y) \subset \sigma(T).$$

A useful criterion for an invariant subspace to be a ν-space, with the proof [Sc.1959, Theorem 4] invariably valid in the unbounded case, is expressed by

4.2. PROPOSITION. Given T with $\rho(T) \neq \emptyset$, $Y \in \text{Inv } T$ is a ν-space of T iff $R(\lambda;T)Y \subset Y$ for all $\lambda \in \rho(T)$.

4.3. COROLLARY. Every hyperinvariant subspace under T is a ν-space of T.

PROOF. If $\rho(T) = \emptyset$, then every invariant subspace is a ν-space of T. Assume that $\rho(T) \neq \emptyset$ and let Y be a hyperinvariant subspace. Since, for each $\lambda \in \rho(T)$, $R(\lambda;T)$ commutes with T, the hypothesis on Y implies that $R(\lambda;T)Y \subset Y$. Thus, Y is a ν-space of T, by 4.2. □

4.4. COROLLARY. Given T, let Y be hyperinvariant under T and suppose that $\sigma(T|Y)$ is compact. Then, the subspaces $\mathcal{T}$ and W, as defined in 3.9, are hyperinvariant under T.

PROOF. Let $A \in B(X)$ commute with T. Then $Y \in \text{Inv } A$ and $A|Y$ commutes with $T|Y$. Moreover, for each $\lambda \in \rho(T|Y)$, $A|Y$ commutes with $R(\lambda;T|Y)$ and hence $A|Y$ commutes with the projection Q (3.12). Then $\mathcal{T} = QY$ and $W = (I_Y - Q)Y$ are invariant under $A|Y$. Thus, $\mathcal{T}$ and W are invariant under A. □

4.5. PROPOSITION. Let T have the SVEP and suppose that $X = \sum_{i=1}^{n} Y_i$. If each Y_i is a ν-space of T, then $\sigma(T) = \bigcup_{i=1}^{n} \sigma(T|Y_i)$.

PROOF. The hypothesis on the Y_i's implies that $\sigma(T) \supset \bigcup_{i=1}^{n} \sigma(T|Y_i)$. The opposite inclusion follows from 2.6 (I) and 2.7. □

If the Y_i's are the summands of a (weak) spectral decomposition of X by T, then T has a *decomposable spectrum* ([J.1977], [Ho.1982]).

4.6. PROPOSITION. If T has the SVEP and $Y \in \text{Inv } T$, then

$$\sigma(y,T) \subset \sigma(y,T|Y) \quad \text{for all} \quad y \in Y.$$

PROOF. Let $y \in Y$. For every $\lambda \in \rho(y,T|Y)$,

$$(\lambda-T)y_T(\lambda) = (\lambda-T|Y)y_{T|Y}(\lambda) = y$$

and hence $\rho(y,T|Y) \subset \rho(y,T)$. □

4.7. DEFINITION. Let T have the SVEP. Then $Y \in \text{Inv } T$ is said to be a μ-*space* of T if

$$\sigma(y,T) = \sigma(y,T|Y) \quad \text{for all} \quad y \in Y.$$

In view of 4.6, $Y \in \text{Inv } T$ is a μ-space of T iff

(4.1) $$\sigma(y,T) \supset \sigma(y,T|Y) \quad \text{for all} \quad y \in Y.$$

4.8. PROPOSITION. Let T have the SVEP. Then

(i) each μ-space of T is also a ν-space of T;

(ii) $Y \in \text{Inv } T$ is a μ-space of T iff

$$\{y(\lambda) : \lambda \in \rho(y,T),\ y \in Y\} \subset Y.$$

PROOF. (i): Let Y be a μ-space of T. With the help of 2.7, one obtains

$$\sigma(T|Y) = \cup\{\sigma(y,T|Y) : y \in Y\} = \cup\{\sigma(y,T) : y \in Y\}$$
$$\subset \cup\{\sigma(x,T) : x \in X\} = \sigma(T).$$

(ii): First, suppose that Y is a μ-space of T. Then, for all $y \in Y$ and $\lambda \in \rho(y,T) = \rho(y,T|Y)$, one has

$$y(\lambda) = y_T(\lambda) = y_{T|Y}(\lambda) \in Y.$$

Conversely, if for all $y \in Y$ and $\lambda \in \rho(y,T)$ we have $y(\lambda) \in Y$, then

$$(\lambda-T|Y)y(\lambda) = (\lambda-T)y(\lambda) = y$$

and hence $\rho(y,T) \subset \rho(y,T|Y)$. Thus Y is a μ-space of T, by (4.1). □

4.9. PROPOSITION. Let T have the SVEP. Then $Y \in \text{Inv } T$ is a μ-space of T iff, for every closed F,

(4.2) $$Y \cap X(T,F) = Y(T|Y,F).$$

For F closed, $Y(T|Y,F) \subset Y \cap X(T,F)$, by 4.6. If Y is a μ-space of T then, for $y \in Y \cap X(T,F)$, one has $\sigma(y,T|Y) = \sigma(y,T) \subset F$ and hence

$$Y \cap X(T,F) \subset Y(T|Y,F).$$

Conversely, assume that (4.2) holds and let $y \in Y$. Denote $F = \sigma(y,T)$ and obtain $y \in Y \cap X(T,F) = Y(T|Y,F)$. Therefore, $\sigma(y,T|Y) \subset F = \sigma(y,T)$ and hence Y is a μ-space of T, by (4.1). □

4.10. DEFINITION. Given T, $Y \in \text{Inv}\, T$ is called an *analytically invariant* subspace under T if, for any analytic function $f : \omega_f \to \mathcal{D}_T$, the condition $(\lambda-T)f(\lambda) \in Y$ implies that $f(\lambda) \in Y$ on an open $\omega_f \subset \mathbb{C}$.

We write AI(T) for the family of analytically invariant subspaces under T.

4.11. PROPOSITION. Every analytically invariant subspace Y under T is a ν-space of T. If, in addition T has the SVEP, then Y is a μ-space of T.

PROOF. Let $Y \in AI(T)$ and $y \in Y$. Since $y = (\lambda-T)R(\lambda;T)y \in Y$ on $\rho(T)$, $R(\lambda;T)y \in Y$ on $\rho(T)$ and hence Y is a ν-space of T, by 4.2. Moreover, if T has the SVEP then, for $y \in Y$ and $\lambda \in \rho(y,T)$, $(\lambda-T)y(\lambda) = y$ implies that $y(\lambda) \in Y$. Thus Y is a μ-space of T, by 4.8 (ii). □

4.12. PROPOSITION. Given T, let $Y \in AI(T)$ be such that $\sigma(T|Y)$ is compact. Then $\mathcal{T} \in AI(T)$ and, if T has the SVEP then $W \in AI(T)$, where $\mathcal{T}$, W were defined by 3.9.

PROOF. Let $f : \omega_f \to \mathcal{D}_T$ be analytic and satisfy condition

$$(4.3) \qquad (\lambda-T)f(\lambda) \in \mathcal{T} \text{ on an open } \omega_f \subset \mathbb{C}.$$

Since $\mathcal{T} \subset Y$ and $Y \in AI(T)$, (4.3) implies that $f(\lambda) \in Y$ on ω_f. In view of 3.11, there are analytic functions $f_1 : \omega \to \mathcal{T}$, $f_2 : \omega \to W$ such that

$$f(\lambda) = f_1(\lambda) + f_2(\lambda) \text{ on an open } \omega \subset \omega_f.$$

Since $f(\omega) \subset \mathcal{D}_T$ and $f_1(\omega) \subset \mathcal{T} \subset \mathcal{D}_T$, it follows that $f_2(\omega) \subset \mathcal{D}_T$. Then (4.3) implies

$$(\lambda-T)f_1(\lambda) \in \mathcal{T}, \quad \lambda \in \omega;$$

$$(4.4) \qquad (\lambda-T)f_2(\lambda) = 0, \quad \lambda \in \omega.$$

Since $\sigma(T|W) = \emptyset$, it follows from (4.4) that $f_2(\lambda) \equiv 0$ and hence $f(\lambda) = f_1(\lambda)$ on ω. Thus $f(\lambda) \in \mathcal{T}$ on ω_f, by analytic continuation. The proof of the second assertion of the proposition is left to the reader. □

4.13. PROPOSITION. Given $T \in B(X)$, suppose that $\sigma(T)$ is nowhere dense and does not separate the plane. Then, every $Y \in \text{Inv}\, T$ is analytically invariant under T.

PROOF. Let $f : \omega_f \to X$ be analytic and satisfy condition

$$h(\lambda) = (\lambda - T)f(\lambda) \in Y \quad \text{on an open } \omega_f \subset \mathbb{C}.$$

Without loss of generality, we may assume that ω_f is connected. Since $\sigma(T)$ is nowhere dense, $\omega_f \cap \rho(T) \neq \emptyset$. Since $\sigma(T)$ does not separate the plane, Y is a ν-space of T. Then, for $\lambda \in \omega_f \cap \rho(T)$, 4.2 implies that $f(\lambda) = R(\lambda;T)h(\lambda) \in Y$ and hence $f(\lambda) \in Y$ on ω_f, by analytic continuation. □

Next, we extend a useful property [Fr.1973, Theorem 1] of analytically invariant subspaces to unbounded closed operators.

4.14. PROPOSITION. Given T, let $Y \in \text{Inv } T$ be such that $Y \subset \mathcal{D}_T$. Y is analytically invariant under T iff $\hat{T} = T/Y$ has the SVEP.

PROOF. First, assume that $\hat{T}$ has the SVEP and let $f : \omega_f \to \mathcal{D}_T$ be analytic and satisfy condition

$$(\lambda - T)f(\lambda) \in Y \quad \text{on an open } \omega_f \subset \mathbb{C}.$$

By the natural homomorphism $X \to X/Y$, we have

$$(\lambda - \hat{T})\hat{f}(\lambda) = \hat{0} \quad \text{on } \omega_f.$$

By the SVEP, $\hat{f}(\lambda) \equiv \hat{0}$ and hence $f(\lambda) \in Y$ for all $\lambda \in \omega_f$.

Conversely, assume that $Y \in AI(T)$. Let $\hat{f} : \omega_{\hat{f}} \to \mathcal{D}_{\hat{T}}$ be analytic and satisfy condition

$$(4.5) \qquad (\lambda - \hat{T})\hat{f}(\lambda) = \hat{0} \quad \text{on an open } \omega_{\hat{f}} \subset \mathbb{C}.$$

Without loss of generality, we assume that $\omega_{\hat{f}}$ is connected. Let

$$\hat{f}(\lambda) = \sum_{n=0}^{\infty} \hat{a}_n (\lambda - \lambda_0)^n$$

be the power series expansion of $\hat{f}$ in a neighborhood δ of $\lambda_0 \in \omega_{\hat{f}}$. For each n, one can choose $a_n \in \hat{a}_n$ such that $\|a_n\| \leq \|\hat{a}_n\| + 1$. Then

$$\limsup_n \|a_n\|^{1/n} \leq \limsup_n \|\hat{a}_n\|^{1/n} + 1$$

and hence

$$f(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n \in \hat{f}(\lambda)$$

is analytic on a neighborhood $\delta' \subset \delta \subset \omega_{\hat{f}}$ of λ_0. Since $\hat{f}(\lambda) \in \mathcal{D}_{\hat{T}}$, there is $h(\lambda) \in \hat{f}(\lambda) \cap \mathcal{D}_T$. Then $\hat{f}(\lambda) = h(\lambda) + Y \subset \mathcal{D}_T$ and hence

$f(\lambda) \in \hat{f}(\lambda) \subset \mathcal{D}_T$, for all $\lambda \in \delta'$. Then (4.5) implies

$$(\lambda-T)f(\lambda) \in Y \quad \text{on } \delta'$$

and by the hypothesis on Y, we have $f(\lambda) \in Y$ on δ'. Thus, $\hat{f}(\lambda) = \hat{0}$ on δ' and hence $\hat{f}(\lambda) = \hat{0}$ on $\omega_{\hat{f}}$, by analytic continuation. □

The following lemma which appeared in [N.1981, Lemma 3.2] has many useful applications.

4.15. LEMMA. Given T, let $Y \in \text{Inv } T$ with $Y \subset \mathcal{D}_T$ be such that $\hat{T} = T/Y$ is closed in X/Y. Suppose that, for $\hat{x} \in X/Y$ and $z \in \mathbb{C}_\infty$, there is a neighborhood V of z and an analytic function $\hat{g} : V \to \mathcal{D}_{\hat{T}}$ satisfying the following condition

$$(\lambda-\hat{T})\hat{g}(\lambda) = \hat{x} \quad \text{for } \lambda \in V \cap \mathbb{C}.$$

Then, there is another neighborhood $V' \subset V$ of z and an analytic function $h : V' \to \mathcal{D}_T$ such that $\hat{h}(\lambda) = \hat{g}(\lambda)$ on V' and $(\lambda-T)h(\lambda)$ is analytic on V'.

PROOF. Let D denote the linear manifold $\mathcal{D}_T$ endowed with the graph norm

$$\|x\|_T = \|x\| + \|Tx\|.$$

T being closed, D is a Banach space and so is D/Y with respect to the usual norm $\|\cdot\|_{D/Y}$ of the quotient space. $D/Y = \mathcal{D}_{\hat{T}}$ can also be endowed with the graph norm $\|\hat{x}\|_{\hat{T}} = \|\hat{x}\| + \|\hat{T}\hat{x}\|$ and since $\hat{T}$ is closed, D/Y is a Banach space with respect to the graph norm $\|\cdot\|_{\hat{T}}$. For any $\hat{x} \in \mathcal{D}_{\hat{T}}$ and all $x \in \hat{x}$, we have

$$\|\hat{x}\|_{\hat{T}} = \|\hat{x}\| + \|\hat{T}\hat{x}\| = \inf_{y\in Y}\|x+y\| + \inf_{w\in Y}\|Tx+w\| \le \inf_{y\in Y}\|x+y\|$$
$$+ \inf_{y\in Y}\|Tx+Ty\| \le \inf_{y\in Y}\{\|x+y\| + \|T(x+y)\|\} = \|\hat{x}\|_{D/Y}.$$

Since D/Y is complete under either norm $\|\cdot\|_{\hat{T}}$, $\|\cdot\|_{D/Y}$, it follows from the open mapping theorem that the two norms are equivalent.

For $\lambda \in V \cap \mathbb{C}$, we have

$$\hat{T}\hat{g}(\lambda) = \lambda\hat{g}(\lambda) - \hat{x}. \tag{4.6}$$

We examine the two possible cases: (a) z is finite and (b) $z = \infty$. In case (a), we may assume that $V \subset \mathbb{C}$. Then $\hat{T}\hat{g}(\cdot)$ is analytic and hence $\hat{g}$ is analytic on V under the norm $\|\cdot\|_{\hat{T}}$ or, equivalently, under the norm $\|\cdot\|_{D/Y}$. By [V.1971, Lemma 2.1], there is a neighborhood $V' \subset V$ of z and an analytic function $h : V' \to D$ such that $\hat{h}(\lambda) = \hat{g}(\lambda)$ on V'. Since $V' \subset \mathbb{C}$ and h is analytic under the norm $\|\cdot\|_T$, $(\lambda-T)h(\lambda)$ is analytic on V'. In case (b), (4.6) rewritten as

$$\hat{T}\,\frac{\hat{g}(\lambda)}{\lambda} = \hat{g}(\lambda) - \frac{\hat{x}}{\lambda},$$

implies that $\hat{g}(\infty) = \hat{0}$. Thus $\lambda\hat{g}(\lambda)$ is analytic and hence so is $\hat{T}\hat{g}(\lambda)$ on V. Consequently, $\hat{g}$ is analytic on V under the norm $\|\cdot\|_{D/Y}$. Since $\hat{g}(\infty) = \hat{0}$, $\hat{g}$ admits the following power series expansion

$$\hat{g}(\lambda) = \sum_{k=1}^{\infty} \hat{a}_k \lambda^{-k} \tag{4.7}$$

in a neighborhood of ∞. Since (4.7) converges in the norm $\|\cdot\|_{D/Y}$, we have

$$\|\hat{a}_k\|_{D/Y} \le M^k \quad \text{for some } M > 0 \text{ and } k \in \mathbb{N}.$$

Thus, there are $a_k \in \hat{a}_k$ such that $\|a_k\|_T \le (M+1)^k$, $k \in \mathbb{N}$ and hence the series

$$h(\lambda) = \sum_{k=1}^{\infty} a_k \lambda^{-k}$$

converges in a neighborhood V' of ∞, under the norm of D. Therefore, h is analytic on V' under the norm of D and $\hat{h}(\lambda) = \hat{g}(\lambda)$ on V'. Consequently, $\lambda h(\lambda)$ is analytic on V' and so is $Th(\lambda)$. □

4.16. PROPOSITION. Given T, let $Y \in \operatorname{Inv} T$ with $Y \subset \mathcal{D}_T$ be such that $\hat{T} = T/Y$ is closed. Then, the following properties hold:

(i) If T has the SVEP and $\sigma(T|Y) \cap \sigma(\hat{T})$ is nowhere dense in $\mathbb{C}$, then $Y \in AI(T)$;

(ii) Let $Z \in \operatorname{Inv} T$ be such that $Y \subset Z \subset \mathcal{D}_T$. Then $Z/Y \in AI(\hat{T})$ iff $Z \in AI(T)$.

PROOF. (i): Let $f : \omega_f \to \mathcal{D}_T$ be analytic such that $(\lambda-T)f(\lambda) \in Y$ on an open $\omega_f \subset \mathbb{C}$. We may assume that ω_f is connected. On the quotient space X/Y we have

$$(\lambda-\hat{T})\hat{f}(\lambda) = \hat{0} \quad \text{on } \omega_f.$$

By 4.15, there is an analytic function $h : \omega_h (\subset \omega_f) \to \mathcal{D}_T$ such that $\hat{h}(\lambda) = \hat{f}(\lambda)$ and $(\lambda-T)h(\lambda)$ is analytic on ω_h. Likewise ω_f, ω_h can be assumed to be a connected open set.

First, suppose that $\omega_h \cap \rho(T|Y) \ne \emptyset$. The function $g : \omega_h \cap \rho(T|Y) \to X$, defined by $g(\lambda) = (\lambda-T)h(\lambda)$, is analytic and

$$\hat{g}(\lambda) = (\lambda-\hat{T})\hat{h}(\lambda) = (\lambda-\hat{T})\hat{f}(\lambda) = \hat{0}$$

implies that $g(\lambda) \in Y$ on $\omega_h \cap \rho(T|Y)$. Then

$$(\lambda-T)[h(\lambda) - R(\lambda;T|Y)g(\lambda)] = 0$$

and by the SVEP,

$$h(\lambda) = R(\lambda;T|Y)g(\lambda) \in Y \text{ on } \omega_h \cap \rho(T|Y). \tag{4.8}$$

Thus $h(\lambda) \in Y$ on ω_h, by analytic continuation. Since $\hat{f}(\lambda)$ and $\hat{h}(\lambda)$ agree on ω_h, $f(\lambda) - h(\lambda) \in Y$ on ω_h. In view of (4.8), $f(\lambda) \in Y$ on ω_f, by analytic continuation.

Next, assume that $\omega_h \subset \sigma(T|Y)$. Since, by hypothesis, $\omega_h \cap \rho(\hat{T}) \neq \emptyset$, it follows from $(\lambda-\hat{T})\hat{h}(\lambda) = \hat{0}$ that $\hat{h}(\lambda) = \hat{0}$ on $\omega_h \cap \rho(\hat{T})$. Thus $\hat{f}(\lambda) = \hat{0}$ on ω_f, by analytic continuation and hence $f(\lambda) \in Y$ on ω_f.

(ii). (Only if): Let $f : \omega_f \to \mathcal{D}_T$ be analytic and satisfy condition

$$(\lambda-T)f(\lambda) \in Z \tag{4.9}$$

on an open connected $\omega_f \subset \mathbb{C}$. On the quotient space X/Y, there corresponds

$$(\lambda-\hat{T})\hat{f}(\lambda) \in Z/Y \text{ on } \omega_f.$$

Then, by hypothesis, $\hat{f}(\lambda) \in Z/Y$ on ω_f or, equivalently, $f(\lambda) \in Z$ on ω_f. Thus Z is analytically invariant under T.

(If): Let $\hat{f} : \omega_{\hat{f}} \to X/Y$ be analytic and satisfy condition

$$(\lambda-\hat{T})\hat{f}(\lambda) \in Z/Y \text{ on an open connected } \omega_{\hat{f}} \subset \mathbb{C}.$$

Fix $\lambda_0 \in \omega_{\hat{f}}$. By an argument used in the second part of the proof of 4.14, $\hat{f}$ can be lifted to a $\mathcal{D}_T$-valued function f, analytic on a neighborhood $\omega \subset \omega_{\hat{f}}$ of λ_0 such that $f(\lambda) \in \hat{f}(\lambda)$ on ω. Then (4.9) holds on ω, $f(\lambda) \in Z$ on ω, $\hat{f}(\lambda) \in Z/Y$ on ω and hence $\hat{f}(\lambda) \in Z/Y$ on $\omega_{\hat{f}}$, by analytic continuation. □

4.17. DEFINITION. Given T, $Y \in \text{Inv}\, T$ is said to be T-*absorbent* if, for any $y \in Y$ and all $\lambda \in \sigma(T|Y)$,

$$(\lambda-T)x = y \tag{4.10}$$

implies that $x \in Y$.

4.18. PROPOSITION. Given T, each T-absorbent space is a ν-space of T.

PROOF. Let Y be a T-absorbent space and suppose that $\sigma(T|Y) \not\subset \sigma(T)$. Then $R(\lambda;T)Y \not\subset Y$ for some $\lambda \in \rho(T) \cap \sigma(T|Y)$ and hence not every solution of (4.10) belongs to Y. This, however, contradicts the definition of Y. □

4.19. PROPOSITION. Given T, let Y be a T-absorbent space. In the following two cases:

(i) $\sigma_p(T) = \emptyset$;

(ii) T has the SVEP, $Y \subset \mathcal{D}_T$ and $\hat{T} = T/Y$ is closed;

Y is analytically invariant under T.

PROOF. Let $f : \omega_f \to \mathcal{D}_T$ be analytic and satisfy condition

$$(\lambda-T)f(\lambda) \in Y \quad \text{on an open} \quad \omega_f \subset \mathbb{C}.$$

Without loss of generality, we assume that ω_f is connected. If $\omega_f \subset \sigma(T|Y)$ then $f(\lambda) \in Y$, by hypothesis. Therefore, assume that $\omega_f \cap \rho(T|Y) \neq \emptyset$. Since

$$g(\lambda) = (\lambda-T)f(\lambda) \in Y \quad \text{on} \quad \omega_f \cap \rho(T|Y),$$

we have

$$(\lambda-T)[f(\lambda) - R(\lambda;T|Y)g(\lambda)] = 0 \quad \text{on} \quad \omega_f \cap \rho(T|Y).$$

In case (i),

$$f(\lambda) = R(\lambda;T|Y)g(\lambda) \in Y \quad \text{on} \quad \omega_f \cap \rho(T|Y)$$

and hence $f(\lambda) \in Y$ on ω_f, by analytic continuation.

In case (ii), use Lemma 4.15 to assert the existence of a function $h : \omega_h \to \mathcal{D}_T$, analytic on an open connected $\omega_h \subset \omega_f$ such that $\hat{h}(\lambda) = \hat{f}(\lambda)$ and $g(\lambda) = (\lambda-T)h(\lambda)$ is analytic on ω_h. On X/Y, we have

$$\hat{g}(\lambda) = (\lambda-\hat{T})\hat{h}(\lambda) = (\lambda-\hat{T})\hat{f}(\lambda) = \hat{0}$$

and hence $g(\lambda) \in Y$ on ω_h. Since Y is T-absorbent, $h(\lambda) \in Y$ on $\omega_h \cap \sigma(T|Y)$. For $\lambda \in \omega_h \cap \rho(T|Y)$, we have

$$(\lambda-T)[h(\lambda) - R(\lambda;T|Y)g(\lambda)] = 0.$$

$R(\lambda;T|Y)g(\lambda)$ being analytic on ω_h, the SVEP of T implies

$$h(\lambda) = R(\lambda;T|Y)g(\lambda) \in Y \quad \text{for } \lambda \in \omega_h \cap \rho(T|Y).$$

Thus $h(\lambda) \in Y$ on all of ω_h and hence $\hat{f}(\lambda) = \hat{h}(\lambda) = \hat{0}$ implies that $f(\lambda) \in Y$ on ω_h and $f(\lambda) \in Y$ on ω_f, by analytic continuation. □

4.20. PROPOSITION. Let T have the SVEP and suppose that

$$X = Y_1 + Y_2.$$

If Y_1 and Y_2 are T-absorbent spaces, then

$$\sigma(T|Y_1 \cap Y_2) \subset \sigma(T|Y_1) \cap \sigma(T|Y_2).$$

PROOF. Let $y \in Y_1 \cap Y_2 = Y$ be arbitrary. Then $R(\lambda;T)y \in Y$ on $\rho(T)$, by 4.2 and 4.18. For $\lambda \in \rho(T|Y_1) \cap \rho(T|Y_2) = \rho(T)$, (where the equality follows from 4.5), we have

$$R(\lambda;T|Y_1)y = [R(\lambda;T)|Y_1]y = R(\lambda;T)y \in Y.$$

Y_2 being T-absorbent, for $\lambda \in \rho(T|Y_1) \cap \sigma(T|Y_2)$, $(\lambda-T)R(\lambda;T|Y_1)y = y$ implies that $R(\lambda;T|Y_1)y \in Y_2$. On the other hand, $R(\lambda;T|Y_1)y \in Y_1$ and hence $R(\lambda;T|Y_1)y \in Y$.

Thus, for all $\lambda \in \rho(T|Y_1)$, we have $R(\lambda;T|Y_1)y \in Y$. Now, 4.2 applied to $Y \in \text{Inv } T|Y_1$, gives $\sigma(T|Y) \subset \sigma(T|Y_1)$. By symmetry, $\sigma(T|Y) \subset \sigma(T|Y_2)$ and the assertion of the proposition follows. □

The property expressed by the foregoing theorem can be extended, via induction, to any finite sum decomposition of X into T-absorbent subspaces.

4.21. PROPOSITION. Given T, let $Y \in \text{Inv } T$ be T-absorbent with $\sigma(T|Y)$ compact. Then $\mathcal{T}$, as defined by 3.9, is T-absorbent.

PROOF. Let $y \in \mathcal{T}$, $\lambda \in \sigma(T|\mathcal{T}) = \sigma(T|Y)$, and let x be a solution of

$$(\lambda-T)x = y. \tag{4.11}$$

Y being T-absorbent, $x \in Y$. There is a representation

$$x = x_0 + x_1 \quad \text{with} \quad x_0 \in \mathcal{T},\ x_1 \in W.$$

By (4.11),

$$(\lambda-T)x_0 = y, \quad (\lambda-T)x_1 = 0,$$

and hence $x_1 = 0$, $x = x_0 \in \mathcal{T}$. □

4.22. DEFINITION. Given T, $Y \in \text{Inv } T$ is said to be a *spectral maximal space* of T if, for any $Z \in \text{Inv } T$, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$.

$Y \in \text{Inv } T$ with $Y \subset \mathcal{D}_T$ is called a T-*bounded spectral maximal space* if conditions $Z \in \text{Inv } T$, $Z \subset \mathcal{D}_T$, $\sigma(T|Z) \subset \sigma(T|Y)$ imply $Z \subset Y$.

We denote by $SM(T)$ and $SM_b(T)$ the family of spectral maximal spaces of T and the family of T-bounded spectral maximal spaces, respectively. Clearly, if $Y \subset \mathcal{D}_T$ is a spectral maximal space of T then Y is a T-bounded spectral maximal space. Conversely, however, not every T-bounded spectral maximal space is a spectral maximal space of T. In fact, if $Y \in SM_b(T)$ and $Z \in \text{Inv } T$ is not contained in $\mathcal{D}_T$, then $\sigma(T|Z) \subset \sigma(T|Y)$ need not imply $Z \subset Y$. For bounded operators, the two concepts coincide.

4.23. PROPOSITION. Given T, every spectral maximal space of T as well as every T-bounded spectral maximal space is hyperinvariant under T.

PROOF. We confine the proof to $Y \in SM(T)$, that of a T-bounded spectral maximal space being similar. Let $A \in B(X)$ commute with T and choose $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|$. Then

$$R(\lambda;A) = \sum_{n=0}^{\infty} \lambda^{-n-1}A^n.$$

For every $x \in \mathcal{D}_T$ and $k \in \mathbb{N}$, we have

$$\sum_{n=0}^{k} \lambda^{-n-1}A^nTx = T(\sum_{n=0}^{k} \lambda^{-n-1}A^nx).$$

T being closed, $k \to \infty$ implies that $R(\lambda;A)x \in \mathcal{D}_T$ and

(4.12) $$R(\lambda;A)Tx = TR(\lambda;A)x.$$

Thus $R(\lambda;A)$ commutes with T. Furthermore, the linear manifold $Y_\lambda = R(\lambda;A)Y$ is closed and hence it is a subspace of X. Evidently,

(4.13) $$R(\lambda;A)(Y \cap \mathcal{D}_T) \subset Y_\lambda \cap \mathcal{D}_T.$$

If $y \in Y_\lambda \cap \mathcal{D}_T$, then $(\lambda-A)y \in Y \cap \mathcal{D}_T$ and $R(\lambda;A)(\lambda-A)y = y$. Therefore, (4.13) is an equality

(4.14) $$R(\lambda;A)(Y \cap \mathcal{D}_T) = Y_\lambda \cap \mathcal{D}_T.$$

Then, for $y \in Y_\lambda \cap \mathcal{D}_T$, there is $x \in Y \cap \mathcal{D}_T$, such that $y = R(\lambda;A)x$. Thus (4.12) and (4.14) imply $Ty = TR(\lambda;A)x = R(\lambda;A)Tx \in Y_\lambda$ and hence $Y_\lambda \in \text{Inv } T$. Moreover, it follows from (4.12) and (4.14) that

$$[R(\lambda;A)]^{-1}(T|Y_\lambda)R(\lambda;A)x = (T|Y)x, \quad x \in Y \cap \mathcal{D}_T.$$

Thus $T|Y_\lambda$ and $T|Y$ are similar and hence

(4.15) $$\sigma(T|Y_\lambda) = \sigma(T|Y).$$

Since $Y \in SM(T)$, (4.15) implies that $Y_\lambda \subset Y$. Consequently, for $|\lambda| > \|A\|$, Y is invariant under $R(\lambda;A)$. It follows from

$$A = \lim_{\lambda\to\infty} \lambda[\lambda R(\lambda;A) - I],$$

that Y is invariant under A. □

4.24. PROPOSITION. Given T, every spectral maximal space of T and each T-bounded spectral maximal space is T-absorbent.