INTRODUCTION

P.J. Webb
Department of Mathematics, The University, Manchester M13 9PL.

This volume presents the core of expository lectures given at the 1985 Durham symposium on representations of algebras. Between them they survey a good part of the cutting edge of research in the area of diagrammatic representation theory, and also with the article by Benson, some recent modular representation theory of groups. The lectures were given to a mixed audience of specialists in various parts of algebra, and it was not possible to assume that everyone was up to the minute with the jargon. An attempt has been made to write down the lectures in a similar spirit in the hope that their readership will be as wide as possible.

It is not easy to get into the area of 'quiver' representation theory from the outside because the area has been expanding so rapidly, and there is no single text where all the fundamentals are developed. Unfortunately the beginner has to read around a bit, and to help with this and set the scene for the articles in this book I propose to give a brief sketch of the way the subject has gone, indicating where accounts of the theory may be found. Useful general references are Gabriel (1980), and Ringel (1984), Chapter 2.

The subject began in the late 1960's with the work of Gabriel, Auslander and several Soviet mathematicians, including Nazarova and Roiter. Gabriel considered representations of quivers because these simultaneously deal with various types of matrix classification and also representations of algebras. See Section 1 of the article by Kraft and Riedtmann for the definitions, also Ringel (1984). Gabriel's theorem that the quivers with only finitely many isomorphism classes of indecomposable modules (= finite representation type) are precisely the Dynkin diagrams led to a conceptual approach in the beautiful paper of Bernstein et al. (1973), where it became evident that finiteness of representation type depends on the positive definiteness of an associated quadratic form, the Tits form (see Kraft and

Riedtmann, this volume), and this is precisely the criterion which distinguishes the Dynkin diagrams from other graphs. This approach also gave rise to a way of constructing all the indecomposable representations of the Dynkin diagrams by means of 'Coxeter functors' or 'reflection functors', so called because they act as reflections on an associated vector space equipped with the Tits form. At this stage the calculations take on a purely combinatorial flavour which leaves the algebra behind, and one has the impression of having got something for nothing. See Gabriel (1980) for these calculations. The direct generalisation of this theory is the identification of the extended Dynkin diagrams as the quivers of tame type and the rest as wild and for this the reader should turn to section 1 of the article by Kraft and Riedtmann. The remainder of that article describes the achievement of Kac in identifying the set of dimension vectors of indecomposable representation of an arbitrary quiver with the positive roots in the corresponding root system (in the sense of Kac-Moody). In reading this it is a great help to have some feel for the properties of the root systems as described in Chapter 5 of Kac (1985).

It is convenient to describe quivers before anything else because the basic notions are quite elementary, involving just linear algebra. But we are really interested in modules over rings, and in fact quiver representations are really the same as modules over an associated algebra, called the path algebra (see 2.1 of Ringel (1985) for this and what follows). This algebra is always hereditary and so is rather special, but an arbitrary finite dimensional algebra is always Morita equivalent to a quotient of a path algebra by an ideal (of relations). Such a set-up consisting of a quiver together with generators for an ideal in the path algebra is called a 'quiver with relations' and this is the approach which will frequently be taken in Ringel's article in this volume. It is a powerful way of describing an algebra and is a means by which the combinatorial methods of quivers are introduced into module categories.

The main link between the combinatorics of quivers and representations of algebras is by means of almost split sequences (which are also called Auslander-Reiten sequences). These came about through Auslander's determination of the simple objects in a certain functor category and the fact that they have resolutions by finitely generated projective functors in many situations (see Gabriel (1980)). At the time this approach was received badly, because it was hard to see the reason for having to go into

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functor categories. To remedy this situation Auslander and Reiten translated the statement about the existence of a resolution of functors to a statement about the existence of sequences of modules, whereupon it achieved a far more widespread success. It must be said, however, that although the simplest way to learn about this for the first time is via the sequences (as in Pierce (1982) for example), the most satisfactory conceptual approach is via the functors. Both are a good idea. In his first article in this volume Auslander surveys the state of the art of the circumstances in which almost split sequences exist. When they do exist, they are short exact sequences of modules $0 \to A \to B \to C \to 0$ which are uniquely specified once either of their end terms A or C is specified. We write $A = \tau C$ and call τ the 'Auslander translate'. Apart from the elegance of their definition, it is hard to see the reason for studying these sequences until certain connections are pointed out. Thus, in the case of path algebras of quivers the Auslander translate is practically the same as one of the Coxeter transformations considered by Bernstein, Gelfand, Ponomarev, the exact relation being given in section 5.4 of Gabriel (1980), and its significance demonstrated later in the same article by constructions of Auslander-Reiten quivers. It became apparent during the seventies that the Auslander-Reiten quiver has some rather simple combinatorial properties and at the same time contains considerable information about the module category, which in the case of finite representation type is more or less complete. There are now many achievements of this approach, and one might cite as examples the approach by Auslander to the first Brauer-Thrall conjecture (proved by Roiter, see Ringel (1980)), the classification by Riedtmann of self-injective algebras of finite representation type (in a series of papers), a variety of other classifications by various authors, the criterion for finite representation type according to Bongartz-Happel-Vossieck (see Ringel's article, these Proceedings), the theorem on the existence of a multiplicative basis in finite representation type algebras by Bautista et al..

A casual glance at pictures of quivers (for example, in this book) reveals that for the most part they seem to consist of regular-looking meshes of arrows which replicate themselves across the page. This is no accident, and depends on the fact that application of the Auslander translate preserves the quiver structure. The best way to view this is via the abstract notion of a translation quiver (Ringel's article), to which

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Riedtmann's Structure Theorem applies (Riedtmann (1980) or Benson (1984), 2.29). It leads to the whole area of covering theory, for which there does not seem to be an adequate self-contained reference. This approach is implicit at points in Ringel's article, and in the work of many authors.

Whereas the Coxeter functors are only defined for hereditary algebras, the Auslander translate has the obvious advantage of working for all finite dimensional algebras. It was perhaps with this in mind that Brenner and Butler (1980) produced a formulation for all algebras of the reflection functors which appear en route in the Coxeter functor definition. They called their new functors tilting functors, and while one can read their original paper it is better to read an account which illustrates their use. For this, lecture 2 of Ringel's article, these Proceedings, will serve.

The article by Knörrer and the second article by Auslander in this book deal (amongst other things) with the representation theory of local rings of singularities. Much recent interest in this derives from the observation by McKay that when G is a finite subgroup of SL(2,C) a graph obtained by the decomposition of tensor products of the irreducible modules is an extension by one point of the desingularisation graph of the corresponding Kleinian singularity (which is defined as the fixed-point ring in the action of G on the power series ring in two variables). For this background, Slodowy (1983) is useful. Finally, in connection with the article of Roiter I would recommend as background reading the remarks of Ringel in lecture 1 of his article here for the connection between poset representations and quiver representations, and also section 2.6 of Ringel (1984). Kleiner's theorem on the posets of finite representation type was originally proved using the 'differentiation process' due to Nazarova and Roiter, and this proof is presented here.

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REPRESENTATION THEORY OF FINITE-DIMENSIONAL ALGEBRAS

DURHAM LECTURES 1985

CLAUS MICHAEL RINGEL
Fakultat fur Mathematik, Universitat Bielefeld,
Universitatsstrasse, D-4800 Bielefeld-1, West Germany.

Let k be an algebraically closed field and A a finite-dimensional k-algebra (associative, with 1). We consider finite-dimensional left A-modules, and call them just modules; the category of all A-modules will be denoted by A-mod. Any module can be written as a (finite) direct sum of indecomposable modules, and the theorem of Krull-Schmidt asserts that such a decomposition is essentially unique: it is unique up to isomorphism. For many purposes it therefore is sufficient to deal only with indecomposable modules. The main problems of the representation theory of finite-dimensional algebras are the following:

- to develop methods for constructing indecomposable modules,
- to look for suitable invariants in order to be able to identify indecomposable modules,
- to show that a given list of indecomposable modules is complete: that it contains a representative of any isomorphism class.

Typical invariants of a module M are the socalled Jordan-Hölder multiplicities: the algebra A has only finitely many simple modules, say $E_1,\dots,E_n$, and we may denote by $(\underline{\dim}\, M)_i$ the multiplicity of E_i occurring in a composition series of M (this is well-known to be an invariant of the isomorphism class of M). The vector $\underline{\dim}\, M$ obtained in this way is called the <u>dimension vector</u> of M. So one may ask for a description of the possible dimension vectors of indecomposable modules for a given algebra, and, having fixed a particular dimension vector, for a description of all indecomposable modules having this dimension vector.

One of the first questions usually will be that about the number of isomorphism classes of indecomposable modules. There may be only finitely many isomorphism classes of indecomposable A-modules, and then A is said to be representation-finite. Examples of representation-finite algebras are first of all the semi-simple ones, but also the algebras of all upper triangular matrices of given size, and there is a vast literature on representation-finite algebras. In case there are infinitely many isomorphism classes of indecomposable A-modules, there are actually always one-parameter families of isomorphism classes of indecomposable A-modules, as was conjectured by Brauer and Thrall. If there exists a two-parameter family of isomorphism classes of indecomposable A-modules, A is said to be wild, otherwise tame. The study of representation-infinite algebras is still in the beginning, only some types of examples seem to be well understood. We will present below several results which are independent of the representation type and exhibit some examples of tame algebras. In addition, we will pose a number of open problems which seem to be worthwhile to study.

A general reference for the terminology used here are our lecture notes [Ri2]. Those notes should also be consulted for the precise attribution of most of the results presented here. Only in case we deal with results which fell out of the scope of [Ri2] or which where not yet available at that time, we will indicate the source. Our aim in these lectures is to give an introduction to the representation theory of finite-dimensional algebras. In particular, we are going to direct the interest towards the main results presented in [Ri2]. In addition, we will report on some recent investigations which are contained in the papers [RV], [Ri3], [Ha] and [HR].

LECTURE 1

THE AUSLANDER-REITEN QUIVER

It will be necessary to consider besides categories of the form A-mod also some related categories, for example full subcategories of A-mod (which are closed under direct sums and direct summands), or the categories of representations of partially ordered sets, or derived categories. Always, the categories which we will deal with will be k-additive categories (thus, additive categories with k operating centrally on the Hom-sets and such that all $\mathrm{Hom}(X,Y)$ are finite-dimensional k-vector-spaces) with split idempotents; and we call such a category a Krull-Schmidt category (note that in a Krull-Schmidt category, any object is a finite direct sum of indecomposable objects, and such a decomposition is unique up to isomorphism).

We start with the basic notions. Given an indecomposable object in a Krull-Schmidt category we call a map $f : X \longrightarrow Y$ a source map for X (the usual name would be "minimal left almost split map") provided the following three conditions are satisfied: first, f is not split mono; second, given any map $f' : X \longrightarrow Y'$ which is not split mono, there is $\eta : Y \longrightarrow Y'$ with $f' = f\eta$; and third, any $\zeta : Y \longrightarrow Y$ with $f = f\zeta$ is an automorphism. There is the following dual notion: Given an indecomposable module Z, we call a map $g : Y \longrightarrow Z$ a sink map for Z (or a "minimal right almost split map") provided, first, g is not split epi; second, given any map $g' : Y' \longrightarrow Z$ which is not split epi, there is η with $g' = \eta g$; and third, any $\zeta : Y \longrightarrow Y$ with $g = \zeta g$ is an automorphism. In case we deal with $\mathcal{K}$ = A-mod where A is a representation-finite algebra, it is not surprising to see that source maps and sink maps exist. The following remarkable theorem asserts that they always do exist in module categories, independent of the representation type:

THEOREM (M. Auslander, I. Reiten). Let A be a finite-dimensional k-algebra. For any indecomposable module M, there exists a source map and a sink map in A-mod, and both are unique up to isomorphism.

Let Z be indecomposable with sink map $g : Y \longrightarrow Z$. Either Z is projective, then we may take for Y the radical $\mathrm{rad}\, Z$ of Z and for g the inclusion map. Or, if Z is not projective, then g is epi, its kernel $\mathrm{Ker}\, g$ is indecomposable and will be denoted by τZ, and the inclusion map $\tau Z \longrightarrow Y$ is a source map.

Dually, let X' be indecomposable with source map $f' : X' \longrightarrow Y'$. Either X' is injective, then we may take $Y' = X'/\operatorname{soc} X'$, and f' the canonical epimorphism. Or, if X' is not injective, then f' is mono, its cokernel $\operatorname{Cok} f'$ is indecomposable and will be denoted by $\tau^- X'$, and the canonical epimorphism $Y' \longrightarrow \tau^- X'$ is a sink map.

Starting with a non-projective indecomposable module Z, or with a non-injective indecomposable module X, we obtain a non-split exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

with f a source map for X, and g a sink map for Z, with $X = \tau Z$ and $Z = \tau^- X$. Sequences of this kind are called Auslander-Reiten sequences. Now, in such an Auslander-Reiten sequence, both modules X, and Z are indecomposable, whereas Y usually is not. We decompose $Y = \bigoplus_i Y_i$, with all Y_i indecomposable, and rewrite the sequence above in the form

$$(*) \qquad 0 \longrightarrow X \xrightarrow{(f_i)_i} \bigoplus_i Y_i \xrightarrow{(g_i)_i} Z \longrightarrow 0 .$$

The maps $f_i : X \longrightarrow Y_i$ are irreducible (we recall the definition below), and we obtain in this way sufficiently many irreducible maps starting in X. The maps $g_i : Y_i \longrightarrow Z$ also are irreducible, and we obtain in this way sufficiently many irreducible maps ending in Z.

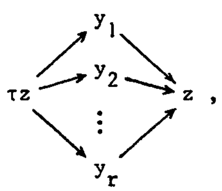
Consider a general Krull-Schmidt category $\mathcal{K}$. If M,M' are indecomposable objects in $\mathcal{K}$, denote by $\operatorname{rad}(M,M')$ the set of non-invertible maps $M \longrightarrow M'$. If M,M' are arbitrary, say with decompositions $M = \bigoplus M_i$, $M' = \bigoplus M'_j$ into indecomposables, let $\operatorname{rad}(M,M') = \bigoplus_{i,j} \operatorname{rad}(M_i,M'_j)$. We obtain in this way an ideal rad in the category $\mathcal{K}$. We define $\operatorname{rad}^d(M,M')$ as the set of maps $M \longrightarrow M'$ which can be written as compositions of d maps all belonging to rad, and let $\operatorname{rad}^\infty = \bigcap_{d\in\mathbb{N}} \operatorname{rad}^d$. If M,M' are indecomposable objects, the maps in $\operatorname{rad}(M,M') \setminus \operatorname{rad}^2(M,M')$ are just the irreducible maps, and the factorspace $\operatorname{Irr}(M,M') = \operatorname{rad}(M,M')/\operatorname{rad}^2(M,M')$ is called the bimodule of irreducible maps.

We have noted above that in the module category A-mod, an Auslander-Reiten sequence (*) displays sufficiently many irreducible maps starting in X or ending in Z. In fact, assume that M is indecomposable, and that $\operatorname{Irr}(X,M)$ is of dimension d. Then, precisely d of the summands Y_i of Y are isomorphic to M, say $Y_1 = \dots = Y_d = M$, and the residue classes of the maps $f_1,\ldots,f_d$ form a basis of $\mathrm{Irr}(X,M)$, whereas the residue classes of the maps $g_1,\ldots,g_d$ form a basis of $\mathrm{Irr}(M,Z)$. In particular, we always have

$$(**) \qquad \dim_k \mathrm{Irr}(\tau Z,M) = \dim_k \mathrm{Irr}(M,Z)$$

for M,Z indecomposable, and Z not projective.

With these preparations, we are going to define the Auslander-Reiten quiver Γ_A of A. We have noted in the introduction that the main object of the representation theory is the study of the set of isomorphism classes of indecomposable modules, and we denote this set by $(\Gamma_A)_o$. We want to endow this set with more structure in order to gain insight into its properties, and the theory expounded above shows that we may consider it as the set of vertices of a socalled translation quiver. Now, a quiver is nothing else than an oriented graph with possible multiple arrows and loops, thus of the form $Q = (Q_o,Q_1,s,e)$, where Q_o,Q_1 are two sets, and $s,e : Q_1 \longrightarrow Q_o$ set maps; the elements of Q_o are called vertices or points, those of Q_1 arrows, and given $\alpha \in Q_1$, then $s(\alpha)$ is called its starting point and $e(\alpha)$ its end point, pictured as follows: $s(\alpha) \xrightarrow{\alpha} e(\alpha)$. A translation quiver Γ is a locally finite quiver with an additional bijection $\tau = \tau_\Gamma : \Gamma'_o \longrightarrow \Gamma''_o$ of two subsets of Γ_o such that for $y \in \Gamma_o$, $z \in \Gamma'_o$, the number of arrows from y to z coincides with the number of arrows from τz to y. (In case we actually fix bijections σ from the set of arrows $y \longrightarrow z$ onto the set of arrows $\tau z \longrightarrow y$, we speak of a polarized translation quiver). A translation quiver may be thought of as being built from small units, the socalled "meshes"



they are defined for any $z \in \Gamma'_o$ (some of the y_i may coincide). The vertices in $\Gamma_o \setminus \Gamma'_o$ are called projective vertices, those in $\Gamma_o \setminus \Gamma''_o$ are called injective vertices. We return to the case of a finite-dimensional algebra A. The isomorphism class of a module M will be denoted by $[M]$. We have already noted that the vertices of Γ_A are of the form