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Lectures on the Asymptotic Theory of Ideals

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INTRODUCTION

In 1982, I was invited to give a course of 11 two-hour lectures in the University of Nagoya on some branch of Commutative Algebra. The topic I chose was the asymptotic theory of ideals and the lectures were duly given between December 1982 and March 1983. The notes below are an extensive revision of the notes given to the audience at the lectures and, with certain exceptions, the chapter headings below correspond to the titles of the individual lectures. The exceptions referred to are the following. First, the notes of the third lecture have been considerably expanded so as to incorporate a proof of the Mori-Nagata Theorem, based on the beautiful theorem of Matijevic, and the original topic of the third lecture, the Valuation Theorem, is dealt with in the fourth lecture. The second change is more considerable. The last three lectures of the course dealt with Teissier's theory of mixed multiplicities as given in Teissier[1973] and was based on the use of complete and joint reductions of a set of ideals. In the last lecture I applied these ideas to prove what I call the general degree formula. An account of the theory of complete and joint reductions has since appeared in Rees[1984], while, since the lectures were given, I have succeeded in proving a still more general degree formula using a quite different method. This method is the method of general elements of ideals and the last three chapters of these notes now deal with the theory of general elements finishing with a proof of the new version of the degree formula. These three chapters are therefore separate from the first nine and can almost be read independently. An earlier version of the material contained in these chapters appeared in Rees[1986].

We now consider the contents of the notes in more detail. The asymptotic theory of ideals originated with the paper Samuel[1952] which contained, in different language, most of the basic ideas lying behind what follows. We commence by considering some of these ideas, using the language of filtrations. By a *filtration* f on a commutative ring A we understand a function defined on A which takes real values, or the value ∞, satisfying the conditions which follow.

i) $f(1) \geq 0$, $f(0) = \infty$; ii) $f(x-y) \geq \mathrm{Min}(f(x),f(y))$; iii) $f(xy) \geq f(x) + f(y)$.

If f,g are filtrations, we consider them as equivalent if there is a finite constant K such that $|f(x)-g(x)| < K$ for all x. This is understood to mean that $g(x) = \infty$ if and only if $f(x) = \infty$. This implies that every filtration is equivalent to a filtration whose values are integers or ∞, and we often restrict attention to such filtrations. One of Samuel's ideas was to introduce a filtration $\mathfrak{f}(x)$ defined as the limit as n tends to ∞ of $f(x^n)/n$. That this limit exists and is a filtration is proved in chapter 2. Samuel's main interest was in the case where A is noetherian and f is a filtration f_J associated with an ideal J and defined by $f_J(x) \geq n$ if and only if x belongs to J^n. Samuel conjectured that $\mathfrak{f}_J(x)$, if finite, was rational. This was proved independently by M. Nagata[1956] and myself[1956b]. It is with the second of these proofs that we are now concerned. A stronger result was proved, referred to in these notes as the Valuation Theorem. This states that $\mathfrak{f}_J(x)$ can be expressed in the form $\text{Min } v_i(x)/v_i(J)$, where v_i ranges over a finite set of valuations v_i, which take non-negative values on A, positive values on J, and the value ∞ on a minimal prime ideal $\mathfrak{p}_i$ of A depending on v_i. The notation $v_i(J)$ denotes the minimal value of $v_i(x)$ on J.

The proof of the theorem depended on the introduction of an ancillary graded ring which we now describe and which in these notes is denoted by $G(f_J)$. This is the sub-ring of $A[t, t^{-1}]$ consisting of all finite sums $\Sigma c_r t^r$, summed, say, from $-p$ to q, satisfying the condition that $f_J(c_r) \geq r$ for each r. Now write u for t^{-1} and G for $G(f_J)$. Then f_J is the restriction to A of the filtration f_{uG} on G and the proof of the Valuation Theorem is reduced to the special case where J is a principal ideal generated by a non-zero divisor. Since G is noetherian, this point being crucial, this special case is a fairly easy deduction from the Mori-Nagata Theorem. In fact the proof of [1956b], obtained in 1955, did not use the Mori-Nagata Theorem, since Nagata's proof of the general case in Nagata[1955] was not then available to me.

The definition of $G(f_J)$ can obviously be adapted to define a graded ring G(f) for any filtration f and the proof indicated above can also be adapted to prove a Valuation Theorem for f, providing that G(f) is noetherian. This leads to the introduction of a class of filtrations, noether filtrations, defined as those for which G(f) is noetherian, with the additional restriction that they take integer values together with ∞. Again a result of Samuel plays a crucial part, his characterisation of graded noetherian rings G as those for which the sub-ring G_0 of elements of degree zero is noetherian and which, in addition, are finitely generated over this sub-ring.

I first met this in Samuel[1953], and it is referred to below as Samuel's Theorem. This theorem enables us to describe noether filtrations in some detail, and this is done in chapter 2. We will merely give one consequence, which appears in chapter 6 as Lemma 6.11, to the effect that, if f is a noether filtration which takes only non-negative values, then it is equivalent to a filtration $w.f_J$ for some ideal J, and some positive integer w. This indicates the key role played by the filtrations f_J.

Now, at last, we consider the individual chapters, restricting ourselves to the first five chapters for the present. Chapter 1 collects together some general results on graded noetherian rings, based for the most part on Samuel's Theorem, but including an account of the theory of Hilbert Functions using the Koszul Complex. Chapter 2 is concerned with elementary results on filtrations, particularly noether filtrations, and we will pick out some of these. First, there is a uniqueness theorem for the representation of $\mathbf{f}(x)$ in the form $\operatorname{Min} v_i(x)/e_i$, where v_i ranges over a finite set of valuations and the numbers e_i are real. Note that the existence of the representation does not appear until chapter 4. Next, in this chapter we associate with a noether filtration f another filtration f* which is integer-valued and closely associated with $\mathbf{f}(x)$. This is the integral closure of f. It is defined by $f^*(x) \geq n$ if x satisfies an equation

$$x^r + c_1 x^{r-1} + \ldots + c_r = 0$$

with $f(c_i) \geq ni$ for each i. f* is related to f, $\mathbf{f}$ by inequalities

$$f(x) \leq f^*(x) \leq \mathbf{f}(x) \leq f^*(x) + 1.$$

In fact $f^*(x)$ is the integral part of $\mathbf{f}(x)$, but this is not proved until chapter 4. Finally, if f and g are two noether filtrations taking only non-negative values, then they are equivalent if and only if $f^*(x) = g^*(x)$ for all x.

Now we come to chapters 3 to 5. The first of these contains a proof of the theorem of Matijevic (Matijevic[1976]) and uses it to prove the Mori-Nagata Theorem (Mori[1952] for local domains and Nagata[1955] for general noetherian domains). The proof given here draws heavily on the papers of Querre[1979] and Kiyek[1981]. Chapter 4 is devoted to a proof of the Valuation Theorem for any noether filtration f. It takes the form

$$\mathfrak{f}(x) = \text{Min}\, v_i(x)/v_i(f)$$

the minimum being taken over a finite set of valuations v_i. This theorem is proved for noether filtrations which may take negative values on A. In this case the definition of $v_i(f)$ is somewhat complicated and we restrict attention here to the case where f(x) takes only non-negative values. First we note that the valuations v_i take values which are non-negative integers or ∞ on A and positive values on the radical of f, this being defined as the set of elements for which $f(x^n) > 0$ for some n. If v_i is proper, that is, takes values other than $0,\infty$, then $v_i(f)$ is the minimum of $v_i(x)/f(x)$ taken over those x for which f(x) is neither 0 or ∞. If v_i is degenerate $v_i(f)$ can be taken to be 1.

Now we come to chapter 5. In this and the later chapters, f is restricted to take non-negative values. Chapter 5 has as its objective the Strong Valuation Theorem. The aim of this theorem is the determination of conditions under which $f^*(x)$ is a noether filtration. It states that this is true for all noether filtrations on A if and only if, for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is analytically unramified, that is, the completion $(A_{\mathfrak{m}})\hat{}$ of $A_{\mathfrak{m}}$ has no nilpotent elements. This requires a great deal of the theory of completions, and hence chapter 5 contains a brief account of the relevant material, given without proofs.

In chapters 6 and 7, the problem considered is that of determining which proper valuations v are associated with some noether filtration f on A via the Valuation Theorem. Since f can be taken to be of the form f_J by Lemma 6.11, these are termed ideal valuations. The two chapters give different characterisations of ideal valuations. In chapter 6, the general case is reduced to the case where A is a local domain $(Q,\mathfrak{m},k,d)$ and v has radical $\mathfrak{m}$. Here $\mathfrak{m}$ is the maximal ideal of Q, $k = Q/\mathfrak{m}$, and d is the Krull dimension of Q. This implies that v has a unique extension $v\hat{}$ to the completion $Q\hat{}$ of Q. Then for v to be an ideal valuation it is necessary and sufficient that this extension $v\hat{}$ takes the value ∞ on a minimal prime ideal $\mathfrak{p}$ of $Q\hat{}$, and that the residue field K_v of v has transcendence degree $\dim(Q\hat{}/\mathfrak{p}) - 1$ over k. Further, given a noether filtration f on Q, the set of valuations associated with f contains one, v say, such that its extension $v\hat{}$ takes the value ∞ on a given minimal prime ideal $\boldsymbol{p}$ of Q^. In chapter 7 it is sufficient to restrict A to be a domain, and for v to be an ideal valuation on A, it is necessary and sufficient for there to exist a finitely generated extension B of A contained in the field of fractions of A with the following two properties: first, that $v(x) \geq 0$ on B and secondly, that the centre $\boldsymbol{p}$ of v on B has height 1. Here, by the centre of v on B, we mean the prime ideal $\boldsymbol{p}$ on which v takes positive values. Using these two characterisations of ideal valuations, we can describe the ideal valuations of a finitely generated extension B of A in terms of those of A. Further, it is possible to use ideal valuations to give new proofs of a number of results of the type of the altitude inequality or concerning chains of prime ideals. This is done in chapter 7.

The next two chapters are concerned with multiplicities. In chapter 8, if f is a noether filtration on a local ring $(Q,\boldsymbol{m},k,d)$ with radical $\boldsymbol{m}$, we associate with f an additive function e(f,M) on the category FG(Q) of finitely generated Q-modules. This is very closely related to the ordinary multiplicity function associated with an ideal. The theory is developed in the short chapter 8 by means of Koszul Complexes. More important from the point of view of these notes is the degree function d(f,M,x) introduced in chapter 9. This depends not only on f, but on an element x of Q satisfying the condition that $\dim(Q/xQ) = d - 1$, and, for fixed f, x is again an additive function of M. It is defined by

$$d(f,M,x) = e(f_x,M/xM) - e(f_x,(0:x)_M)$$

where f_x is the filtration on Q/xQ defined by taking $f_x(y) = \mathrm{Max}(f(y'))$, where y' ranges over the inverse images of y under the map $Q \to Q/xQ$, and M/xM, $(0:x)_M$ are considered as (Q/xQ)-modules. The main object of these two chapters is to prove a degree formula

$$d(f,M,x) = \Sigma\delta(v)L_v(M)d(f,v)v(x),$$

where v ranges over the valuations associated with f such that the residue field of v has transcendence degree d-1, $\delta(v)$ is the length of the primary component of zero in Q^ corresponding to the prime ideal $\boldsymbol{p}$ of Q^ on which v^ takes the value infinity, and $L_v(M)$ is the length of the module $M_{\boldsymbol{p}(v)}$ over the artinian ring $Q_{\boldsymbol{p}(v)}$, where $\boldsymbol{p}(v)$ is the minimal prime ideal of Q on which v takes the value ∞. The numbers d(f,v) are positive rational numbers. Chapter 9 concludes with a proof, using the degree formula, of the theorem that, if Q is quasi-unmixed (that is, $\dim(Q\hat{}/\boldsymbol{p}) = d$ for all minimal primes $\mathbf{p}$ of $Q^{\wedge}$), and f,g are two noether filtrations with radical $\mathbf{m}$ such that $g(x) \geq f(x)$ for all x, then f,g are equivalent if and only if $e(f,Q) = e(g,Q)$.

Now we turn to the last three chapters. We have already remarked that these chapters are separate from the first 9, and that the key idea underlying them is that of general elements of ideals. If $(Q,\mathbf{m},k,d)$ is a local ring and J is an ideal of Q with a basis $a_1,\ldots,a_m$, then a natural way of defining a general element x of J is to take $x = \Sigma X_i a_i$, where $X_1,\ldots,X_m$ are indeterminates over Q. Further, if we localise $Q[X_1,\ldots,X_m]$ at $\mathbf{m}[X_1,\ldots,X_m]$, we keep within the theory of local rings. However, in situations where we have to consider several general elements of either the same or different ideals, this means that we have to adjoin more indeterminates. Eakin and Sathaye[1976] overcame this difficulty by adjoining a countable set of indeterminates $X_1,X_2,\ldots$ and localising at $\mathbf{m}[X_1,X_2,\ldots]$, using the resulting ring with considerable success. It is this idea we follow in these three chapters, the resulting ring being termed the *general extension* of Q and denoted by Q_g. Q_g can also be considered as the union of the rings Q_N obtained by localising $Q[X_1,\ldots,X_N]$ at $\mathbf{m}[X_1,\ldots,X_N]$. In these notes considerable use is made of the fact that Q_g is noetherian. This is a particular case of a general result of Grothendieck which appears as Proposition 1 in the appendix to chapter 9 of [BAC], but an *ad hoc* proof of the fact that Q_g is noetherian is given in chapter 10. Another aspect of Q_g is that it is a regular extension of Q, which implies that any reasonable condition imposed on Q is almost certainly also satisfied by Q_g. This is verified in a number of important cases in chapter 10. However, the main object of chapter 10 is twofold. First we study the relationship between a prime ideal $\mathbf{P}$ of Q_g and the prime ideal $\mathbf{p} = \mathbf{P} \cap Q$ of Q. The basic result is Theorem 10.23 of chapter 10. Secondly we consider $\mathbf{m}$-valuations. An $\mathbf{m}$-valuation v on Q is a valuation non-negative on Q, positive on $\mathbf{m}$, and taking the value ∞ on some prime ideal $\mathbf{p}$ of Q, not necessarily minimal. We require the further restriction on v that its residue field K_v be finitely generated over k. Finally we say that v is *good* if

$$\text{trans.deg}_k K_v = \dim(Q/\mathbf{p}) - 1.$$

The results we require on $\mathfrak{m}$-valuations on Q, Q_g are set out in the third part of chapter 10.

In chapter 11 we now come to general elements. We will only consider here the simplest case, that of a general element of an ideal J. The definition already given is taken here as the definition of a standard general element x of J. A general element x of J is then any element of the form T(x), where T is an automorphism of Q_g over Q. This definition turns out to be independent of the choice of the particular set of generators $a_1,\dots,a_m$ of J we start with. It is not too difficult to extend the same idea to consider independent sets of general elements $x_1,\dots,x_s$ of a set of ideals $J_1,\dots,J_s$ and to remark upon results such as the ideal $Q\cap(x_1Q_g + \dots + x_sQ_g)$ depending only on $J_1,\dots,J_s$.

Now we turn to the final chapter where the object is to generalise the degree formula of chapter 9, although filtrations play no part in the generalisation. We start with a definition. A set of ideals $J_1,\dots,J_r$ of Q is said to be *independent* if a set of independent general elements $x_1,\dots,x_r$ of these ideals is a sub-set of a set of parameters of Q_g. If r = d, then we can define a mixed multiplicity $e(Q|\,\mathbf{J}\,|M)$ as the multiplicity $e(Q_g|\,x_1,\dots,x_d\,|M\otimes_Q Q_g)$, as introduced by D.J. Wright and described in detail in Northcott[LRMM]. This depends only on the set of ideals $\mathbf{J} = (J_1,\dots,J_d)$, and not on the set of independent general elements $x_1,\dots,x_d$ chosen. If the ideals of $\mathbf{J}$ are all $\mathfrak{m}$-primary, this is Teissier's mixed multiplicity, but it is defined for some other sets of ideals $\mathbf{J}$. The function $e(Q|\,\mathbf{J}\,|M)$ is symmetric in the ideals $J_1,\dots,J_d$, takes non-negative integer values and has the nice property that, if we write $\mathbf{J}'$ for the set $(J_1,\dots,J_{d-1})$,

$$e(Q|\,\mathbf{J}',J_dK_d\,|M) = e(Q|\,\mathbf{J}',J_d\,|M) + e(Q|\,\mathbf{J}',K_d\,|M).$$

The general degree formula now arises as follows. We consider a set of d-1 independent ideals $\mathbf{J}'$ as above. If J is any $\mathfrak{m}$-primary ideal, the set $(\mathbf{J}',J)$ is also independent, and hence for any M, $e(Q|\,\mathbf{J}',J\,|M)$ can be considered as a function on the set of $\mathfrak{m}$-primary ideals of Q with non-negative integer values. In its simplest form, the degree formula is a formula

$$e(Q|\,J',J\,|M) = \Sigma a(J';M;v)v(J)$$

the sum being over all good $\mathfrak{m}$-valuations on Q. The existence of such a formula implies the uniqueness of the coefficients a(J';M;v) as a result of a theorem proved in chapter 10. We refer the reader to chapter 12 for more specific information concerning the coefficients a(J';M;v). We simply note here that the values of a(J';M;v) are non-negative integers, and, as a function of M, they are additive functions on FG(Q).