THE LOGARITHMIC INTEGRAL
II

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$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\,dt$$

The logarithmic integral

II

PAUL KOOSIS
McGill University in Montreal

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Paul Koosis
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Foreword to volume II, *with an example for the end of volume* I

Art is long and life is short. More than four years elapsed between completion of the MS for volume I and its publication; a good deal of that time was taken up with the many tasks, often tedious, called for by the production of any decently printed book on mathematics.

An attempt has been made to speed up the process for volume II. Three quarters of it has been set directly from handwritten MS, with omission of the intermediate preparation of typed copy, so useful for bringing to light mistakes of all kinds. I have tried to detect such deficiencies on the galleys and corrected all the ones I could find there; I hope the result is satisfactory.

Some mistakes did remain in volume I in spite of my efforts to remove them; others crept in during the successive proof revisions. Those that have come to my attention are reported in the *errata* immediately following this foreword.

In volume I the theorem on simultaneous polynomial approximation was incorrectly ascribed to Volberg; it is almost certainly due to T. Kriete, who published it some three years earlier. L. de Branges' name should have been mentioned in connection with the theorem on p. 215, for he gave (with different proof) an essentially equivalent result in 1959. The developments in §§A and C of Chapter VIII have been influenced by earlier work of Akhiezer and Levin. A beautiful paper of theirs made a strong impression on me many years ago. For exact references, see the bibliography at the end of this volume.

I thank Jal Choksi, my friend and colleague, for having frequently helped me to extricate myself from entanglements with the English language while I was writing and revising both volumes.

Suzanne Gervais, maker of animated films, became my friend at a bad time in my life and has constantly encouraged me in my work on this book,

from the time I first decided I would write it early in 1983. Although she had visual work enough of her own to think about, she was always willing to examine my drawings of the figures and give me practical advice on how to do them. For that help and for her friendship which I am fortunate to enjoy, I thank her affectionately.

One point raised at the very end of volume I had there to be left unsettled. This concerned the likelihood that Brennan's improvement of Volberg's theorem, presented in article 1 of the addendum, was essentially best possible. An argument to support that claim was made on pp. 578–83; it depended, however, on an example which had been reported, but not described, by Borichev and Volberg. No description was available before Volume I went to press, so the claim about Brennan's improvement could not be fully substantiated.

Now we are able to complete verification of the claim by providing the missing example. Its description is found at the end of a paper by Borichev and Volberg appearing in the very first issue of the new Leningrad periodical *Algebra i analiz*. We continue using the notation of the addendum to volume I.

Two functions have to be constructed. The first, $h(\xi)$, should be decreasing for $0 < \xi < \infty$ and satisfy $\xi h(\xi) \geqslant 1$, together with the relation

$$\int_0^1 \log h(\xi)\,d\xi \quad = \quad \infty.$$

The second, $F(z)$, is to be continuous on the closed unit disk and $\mathscr{C}_\infty$ in its interior, with

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \quad \leqslant \quad \exp(-h(\log(1/|z|))), \qquad |z| \; < \; 1,$$

$$|F(e^{i\vartheta})| \quad > \quad 0 \qquad \text{a.e.},$$

and

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \quad = \quad -\infty.$$

The function $F(z)$ we obtain will in fact be analytic in most of the unit disk Δ, ceasing to be so only in the neighborhood of some very small segments on the positive radius, accumulating at 1. The function $h(\log(1/x))$ will be very much larger than $1/\log(1/x)$ for most of the

$x \in (0,\ 1)$ contiguous to those segments.

Three simple ideas form the basis for the entire construction:

(1) In a domain $\mathscr{E}$ with piecewise analytic boundary having a 90° corner (internal measure) at ζ, say, we have

$$\omega_{\mathscr{E}}(I,\ z) \quad \leqslant \quad K_z|I|^2, \qquad z \in \mathscr{E},$$

for arcs I on $\partial\mathscr{E}$ containing ζ (see volume I, pp 260–1);

(2) The use of a Blaschke product involving factors affected with fractional exponents to 'correct', in an infinitely connected subdomain of Δ, a function analytic there and multiple-valued, but with single valued modulus;

(3) The use of a smoothing operation inside Δ, scaled according to that disk's hyperbolic geometry.

We start by looking at harmonic measure in domains $\mathscr{E} = \Delta \sim [a,\ 1]$, where $0 < a < 1$. According to (1), if $\eta > 0$ is small (and $< 1-a$), we have

$$\omega_{\mathscr{E}}(E_\eta,\ 0) \quad \leqslant \quad O(\eta^2)$$

for the sets $E_\eta = [1-\eta,\ 1] \cup I_\eta$, where I_η is the arc of length η on the unit circle, centered at 1. This is so because $\partial\mathscr{E}$ has two (internal) square corners at 1 that contribute separately to harmonic measure. (The slit $[a,\ 1]$ can be opened up by making a conformal mapping of $\mathscr{E}$ given by $z \longrightarrow \sqrt{(a-z)}$; when this is done the two corners at 1 are separated and they remain square.) Suppose that, for some given $a \in (0,\ 1)$, we fix an $\eta > 0$ small enough to make $\omega_{\mathscr{E}}(E_\eta,\ 0)/\eta$ less than some preassigned amount. Then, if we put $\mathscr{G} = \Delta \sim [a,\ 1-\eta]$, we will have, by simple comparison of $\omega_{\mathscr{G}}(I_\eta,\ z)$ and $\omega_{\mathscr{E}}(E_\eta,\ z)$ in $\mathscr{E}$,

$$\omega_{\mathscr{G}}(I_\eta,\ 0)/\eta \quad \leqslant \quad \omega_{\mathscr{E}}(E_\eta,\ 0)/\eta.$$

This relation is taken as the base of an inductive process. Beginning with an $a_1 > 2/(2+\sqrt{3})$ and < 1 (we shall see presently why the first condition is needed), we take a b_1, $a_1 < b_1 < 1$, so close to 1 as to make

$$\frac{\omega_{\mathscr{G}_1}(I_1,\ 0)}{|I_1|} \quad < \quad \frac{1}{2}$$

for $\mathscr{G}_1 = \Delta \sim [a_1,\ b_1]$ and the arc I_1 of length $1-b_1$ on $\partial\Delta$ centered at 1. One next chooses a_2, $b_1 < a_2 < 1$, in a way to be specified later on (a_2 will in fact be much *closer* to 1 than b_1), and then takes

b_2, $a_2 < b_2 < 1$, near enough to 1 to have

$$\frac{\omega_{\mathscr{G}_2}(I_2, 0)}{|I_2|} < \frac{1}{4}$$

and

$$|I_2| < \tfrac{1}{2}|I_1|$$

for $\mathscr{G}_2 = \Delta \sim [a_2, b_2]$ and the arc I_2 of length $1-b_2$ on $\partial\Delta$ centered at 1. Continuing this procedure indefinitely, we get a sequence of segments

$$J_n = [a_n, b_n],$$

where $b_n < a_{n+1} < b_{n+1} < 1$, and nested arcs I_n of length $1-b_n$ on $\partial\Delta$, each centered at 1, with

$$\frac{\omega_{\mathscr{G}_n}(I_n, 0)}{|I_n|} < \frac{1}{2^n}$$

for the corresponding domains $\mathscr{G}_n = \Delta \sim J_n$, and

$$|I_n| < \tfrac{1}{2}|I_{n-1}|.$$

Take now

$$\mathscr{D} = \Delta \sim J_1 \sim J_2 \sim J_3 \sim \cdot \cdot \cdot ;$$

then, since $\mathscr{D}$ is contained in each $\mathscr{G}_n$, the principle of extension of domain tells us that

$$\frac{\omega_{\mathscr{D}}(I_n, 0)}{|I_n|} \leqslant \frac{\omega_{\mathscr{G}_n}(I_n, 0)}{|I_n|} < \frac{1}{2^n}.$$

Our first ingredient in the formation of the desired $F(z)$ is a function $u(z)$ positive and harmonic in $\mathscr{D}$. Let $T_n(\vartheta)$ be periodic of period 2π, with

$$T_n(\vartheta) = \frac{1}{|I_n|}\left(1 - \frac{2|\vartheta|}{|I_n|}\right)^+ \quad \text{for } -\pi \leqslant \vartheta \leqslant \pi.$$

The graph of $T_n(\vartheta)$ for $|\vartheta| \leqslant \pi$ is an isosceles triangle of height $1/|I_n|$ with its base on the segment $\{|\vartheta| \leqslant |I_n|/2\}$ corresponding to the arc I_n. We have

$$\int_{-\pi}^{\pi} T_n(\vartheta)\,d\vartheta = \frac{1}{2}$$

while

$$\int_{-\pi}^{\pi} T_n(\vartheta)\, d\omega_{\mathscr{D}}(e^{i\vartheta}, 0) \quad \leqslant \quad \frac{\omega_{\mathscr{D}}(I_n, 0)}{|I_n|} \quad < \quad \frac{1}{2^n},$$

so

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\ d\omega_{\mathscr{D}}(e^{i\vartheta}, 0) \quad < \quad \infty.$$

although

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\ d\vartheta \quad = \quad \infty$$

For $z \in \mathscr{D}$, we put

$$u(z) \quad = \quad \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\ d\omega_{\mathscr{D}}(e^{i\vartheta}, z)\ ;$$

the integral on the right is certainly *finite* by the third of the preceding four relations and Harnack's inequality, so $u(z)$ is harmonic in $\mathscr{D}$ and

$$u(z) \ > \ 0$$

there. For $0 < |\vartheta| \leqslant \pi$, $\sum_{n=1}^{\infty} T_n(\vartheta)$ is continuous (and *even* locally Lip 1 !), so at these values of ϑ,

$$u(z) \quad \longrightarrow \quad \sum_{n=1}^{\infty} T_n(\vartheta) \qquad \text{as } z \longrightarrow e^{i\vartheta}$$

from within $\mathscr{D}$. (It is practically obvious that the corresponding points $e^{i\vartheta}$ are regular for the Dirichlet problem in $\mathscr{D}$ – in fact, *all* points of $\partial\mathscr{D}$ are regular.) Taking $u(e^{i\vartheta})$ *equal* to $\sum_{n=1}^{\infty} T_n(\vartheta)$, we thus get a function $u(z)$ continuous in $\bar{\Delta} \sim \{1\}$, and we have

$$\int_{-\pi}^{\pi} u(e^{i\vartheta})\, d\vartheta \quad = \quad \infty.$$

The function $u(z)$ has, *locally*, a harmonic conjugate $\tilde{u}(z)$ in $\mathscr{D}$. The latter, of course, need not be *single-valued* in the infinitely connected domain $\mathscr{D}$; we nevertheless put

$$f(z) \quad = \quad e^{-(u(z)+i\tilde{u}(z))}$$

for $z \in \mathscr{D}$, obtaining a function *analytic and multiple-valued* in $\mathscr{D}$ whose *modulus*, $e^{-u(z)}$, is *single-valued* there. If $e^{i\vartheta} \neq 1$, *any given branch* of $\tilde{u}(z)$ is *continuous up to* $e^{i\vartheta}$, because $u(e^{it}) = \sum_{n=1}^{\infty} T_n(t)$ is Lip 1 for t near ϑ.

(To verify this, it suffices to look at $u(z)$ and $\tilde{u}(z)$ in the intersection of $\mathscr{D}$ with a small disk about $e^{i\vartheta}$ avoiding the J_n; if there is still any doubt, map that intersection conformally onto Δ.) It therefore makes sense to talk about the *multiple-valued*, but locally continuous boundary value $f(e^{i\vartheta})$ when $e^{i\vartheta} \neq 1$; the modulus $|f(e^{i\vartheta})|$ is again single-valued, being equal to $\exp(-u(e^{i\vartheta}))$. By the previous relation, we have

$$\int_{-\pi}^{\pi} \log|f(e^{i\vartheta})|d\vartheta = -\infty.$$

It is now necessary to cure the multiple-valuedness of $f(z)$; that is where the second of our ideas comes in. In constructing the $J_n = [a_n, b_n]$ and the arcs I_n, there is nothing to prevent our choosing the a_n so as to have

$$\sum_{n=1}^{\infty} (1 - a_n) < \infty;$$

we henceforth assume that this has been done. (A much *faster* convergence of a_n to 1 will indeed be required later on.) Our condition on the a_n guarantees that the sum

$$\sum_{n=1}^{\infty} \mu_n \log\left|\frac{z - a_n}{1 - a_n z}\right|$$

converges uniformly in the interior of $\Delta \sim \bigcup_n \{a_n\} \supseteq \mathscr{D}$ whenever the coefficients μ_n are bounded. If $0 \leqslant \mu_n \leqslant 1$, that sum is then equal to a function $v(z)$, *harmonic and* $\leqslant 0$ in $\mathscr{D}$. For the latter, there is a multiple-valued harmonic conjugate $\tilde{v}(z)$ defined in $\mathscr{D}$, and we have finally a function

$$b(z) = e^{v(z) + i\tilde{v}(z)} = \prod_{n=1}^{\infty} \left(\frac{a_n - z}{1 - a_n z}\right)^{\mu_n},$$

analytic but multiple-valued in $\mathscr{D}$. The *modulus* $|b(z)| = e^{v(z)}$ is single-valued in $\mathscr{D}$.

The points a_n accumulate only at 1, so any branch of $b(z)$ is continuous up to points $e^{i\vartheta} \neq 1$ of the unit circle. For such points, $|b(e^{i\vartheta})| = 1$, and of course $|b(z)| \leqslant 1$ in $\mathscr{D}$, since $v(z) \leqslant 0$ there.

By proper adjustment of the exponents μ_n we can make the *product* $b(z)f(z)$ *single-valued* in $\mathscr{D}$, and hence analytic there in the ordinary sense. Consider what happens when z describes a simple closed path in the counterclockwise sense about just one of the slits $J_n = [a_n, b_n]$. Each given branch of the harmonic conjugate $\tilde{u}(z)$ will then increase by a certain (real) amount λ_n, independent of the branch. At the same time, every branch of $\tilde{v}(z) = \arg b(z)$ will increase by $2\pi\mu_n$. We take μ_n between 0 and 1 so as to make

$$2\pi\mu_n - \lambda_n$$

an integral multiple of 2π; this is clearly possible, and, once it is done, every branch of $\arg(b(z)f(z)) = \tilde{v}(z) - \tilde{u}(z)$ *increases by that amount* when z goes around a path of the kind just mentioned. Then the *product* $b(z)f(z)$ just comes back to its original value! Choosing in this way a value of μ_n, $0 \leqslant \mu_n \leqslant 1$, for *every* n, we ensure that $b(z)f(z)$ is single-valued in $\mathscr{D}$. Note that we have

$$|b(z)f(z)| \leqslant e^{-u(z)} \leqslant 1, \qquad z \in \mathscr{D},$$

and, since $|b(e^{i\vartheta})| = 1$ for $e^{i\vartheta} \neq 1$,

$$|b(e^{i\vartheta})f(e^{i\vartheta})| = |f(e^{i\vartheta})| > 0, \qquad e^{i\vartheta} \neq 1.$$

Because the product $b(z)f(z)$ is analytic in $\mathscr{D}$, we have there

$$\frac{\partial}{\partial\bar{z}}\, b(z)f(z) = 0;$$

the expression on the left may therefore be looked on as a *distribution* in Δ, supported on the slits J_n of $\Delta \sim \mathscr{D}$. In order to obtain a $\mathscr{C}_\infty$ function defined in Δ, we *smooth out* $b(z)f(z)$; that is our *third idea*. The smoothing is scaled according to the *square* of the gauge for the hyperbolic metric in Δ, i.e., like $1/(1-|z|)^2$.

Taking a $\mathscr{C}_\infty$ function $\psi(\rho) \geqslant 0$ supported on the interval $[1/4, 1/2]$ of the real axis, with

$$\int_0^{1/2} \psi(\rho)\rho\, d\rho = \frac{1}{2\pi},$$

we put, for $z \in \Delta$,

$$G(z) = \iint_\Delta \psi\left(\frac{|z-\zeta|}{(1-|z|)^2}\right) \frac{b(\zeta)f(\zeta)}{(1-|z|)^4}\, d\xi\, d\eta$$

(writing, as usual, $\zeta = \xi + i\eta$).

The first thing to observe here is that the expression on the right *makes sense*. Although $b(\zeta)f(\zeta)$ is defined merely in $\mathscr{D}$, the slits J_n making up $\Delta \sim \mathscr{D}$ are of *planar Lebesgue measure zero*, so we *only need* the values of the product in $\mathscr{D}$ in order to do the integral. The *second* observation is that $G(z)$ is $\mathscr{C}_\infty$ in $\mathscr{D}$. As a function of ζ, $\psi(|z-\zeta|/(1-|z|)^2)$ vanishes outside the disk $|\zeta - z| \leqslant \frac{1}{2}(1-|z|)^2$ which, however, lies well within Δ

for $z \in \Delta$, since then $|z| + \frac{1}{2}(1-|z|)^2 < 1$. We may therefore differentiate under the integral sign with respect to z or $\bar{z}$ as often as we wish, $\psi(\rho)$ being $\mathscr{C}_\infty$ (its identical vanishing for ρ near 0 helps here), and $|b(\zeta)f(\zeta)|$ being < 1 in $\mathscr{D}$. In this way we verify that $G(z)$ is $\mathscr{C}_\infty$ in Δ, and get (practically 'by inspection') the crude estimate

$$\left|\frac{\partial G(z)}{\partial \bar{z}}\right| \leqslant \frac{\text{const.}}{(1-|z|)^2}, \qquad |z| < 1.$$

As for $G(z)$, just *an average* of the function $b(\zeta)f(\zeta)$, we have

$$|G(z)| < 1, \qquad |z| < 1.$$

The *third* thing to observe is that $G(z)$ is actually *analytic* in a fairly large subset of Δ. Because $\psi(\rho)$ vanishes for $\rho \geqslant 1/2$, the integration in the above formula for $G(z)$ is really over the disk

$$\bar{\Delta}_z = \{\zeta: \ |\zeta - z| \leqslant \tfrac{1}{2}(1-|z|)^2\}$$

which, as we have just seen, *lies in* Δ when $|z| < 1$. Suppose that $\bar{\Delta}_z$ touches *none* of the slits J_n. Then $\bar{\Delta}_z \subseteq \mathscr{D}$ where $b(\zeta)f(\zeta)$ is *analytic* and, writing $\zeta = z + re^{i\vartheta}$, we have

$$G(z) = \int_0^{(1-|z|)^2/2}\int_{-\pi}^{\pi} b(z+re^{i\vartheta})f(z+re^{i\vartheta})\psi(r/(1-|z|)^2)\frac{r\,d\vartheta\,dr}{(1-|z|)^4}.$$

Using Cauchy's theorem to perform the first integration with respect to ϑ and then making the change of variable $r/(1-|z|)^2 = \rho$, we obtain the value $2\pi b(z)f(z)\int_0^{1/2}\psi(\rho)\rho\,d\rho = b(z)f(z)$, i.e.,

$$G(z) = b(z)f(z) \qquad \text{if} \quad \bar{\Delta}_z \subseteq \mathscr{D}.$$

When $\bar{\Delta}_z \subseteq \mathscr{D}$, the disks $\bar{\Delta}_{z'}$ also lie in $\mathscr{D}$ for the z' belonging to some *neighborhood* of z; we thus have $G(z') = b(z')f(z')$ in that neighborhood, and $G(z')$ (like $b(z')f(z')$) is then *analytic at* z. For the z in Δ such that $\bar{\Delta}_z \subseteq \mathscr{D}$, we therefore have

$$\frac{\partial G(z)}{\partial \bar{z}} = 0$$

although, for the *remaining* z in the unit disk, only the above estimate on $\partial G(z)/\partial\bar{z}$ is available. It is necessary to examine the set of those remaining z.

They are precisely the ones for which $\bar{\Delta}_z$ intersects with some J_n. We proceed to describe the set

$$B_n = \{z \in \Delta: \ \bar{\Delta}_z \cap J_n \neq \varnothing\}.$$

Write for the moment $J_n = [a, b]$, dropping the subscripts on a_n and b_n. If $\bar{\Delta}_z$ is to intersect with $[a, b]$, we must have $|z| > 2-\sqrt{3}$. Indeed, a, as one of the a_n, is $\geqslant a_1$ which we initially took $> 2/(2+\sqrt{3})$, while $\bar{\Delta}_z$ lies in the disk $\{|\zeta| \leqslant |z| + \frac{1}{2}(1-|z|)^2\}$ whose radius increases with $|z|$. For $|z| = 2-\sqrt{3}$, that radius works out to $2/(2+\sqrt{3})$, so if $|z| \leqslant 2-\sqrt{3}$, $[a, b]$ would lie *outside* the disk containing $\bar{\Delta}_z$; $|z|$ is thus $> 2-\sqrt{3}$ for $z \in B_n$. Now when $2-\sqrt{3} < |z| < 1$, $|z| - \frac{1}{2}(1-|z|)^2 > 0$, so $\bar{\Delta}_z$ is in fact contained in the *ring*

$$|z| - \tfrac{1}{2}(1-|z|)^2 \leqslant |\zeta| \leqslant |z| + \tfrac{1}{2}(1-|z|)^2$$

(*that's* why a_1 was chosen $> 2/(2+\sqrt{3})$!). Therefore, if $\bar{\Delta}_z$ intersects with $[a, b]$, we must have

$$|z| - \tfrac{1}{2}(1-|z|)^2 \leqslant b,$$
$$|z| + \tfrac{1}{2}(1-|z|)^2 \geqslant a.$$

Both left sides are increasing functions of $|z|$ (for $z \in \Delta$), so these relations are equivalent to

$$a' \leqslant |z| \leqslant b',$$

where

$$a' + \tfrac{1}{2}(1-a')^2 = a,$$
$$b' - \tfrac{1}{2}(1-b')^2 = b.$$

In $(0, 1)$ these equations have the solutions

$$a' = \sqrt{(2a-1)},$$
$$b' = 2-\sqrt{(3-2b)};$$

for the first we need $a > 1/2$ but have in fact $a > 2/(2+\sqrt{3})$. Using differentiation, one readily verifies that $a' < a$ and $b < b' < 1$.

We see that B_n (the set of $z \in \Delta$ for which $\bar{\Delta}_z$ intersects with $[a, b]$) is an oval-shaped region including $[a, b]$ and contained in the ring $a' \leqslant |z| \leqslant b'$; its boundary crosses the x-axis at the points a' and b'. When a is close to 1, B_n is quite thin in the vertical direction because, if $\bar{\Delta}_z$ touches the x-axis at all, we must have $|\Im z| \leqslant \frac{1}{2}(1-|z|)^2$.

One can specify the a_n and b_n so as to ensure *disjointness* of the oval regions B_n. The preceding description shows that this will be the case if the *rings* $a'_n \leqslant |z| \leqslant b'_n$ are disjoint, where (restoring the subscript n)

$$a'_n = \sqrt{(2a_n-1)},$$
$$b'_n = 2 - \sqrt{(3-2b_n)};$$

i.e., if $b'_n < a'_{n+1}$ for $n = 1, 2, 3, \ldots$. It is easy to arrange this in making the successive choices of the a_n and b_n; all we need is to have

$$a_{n+1} \;=\; a'_{n+1} \;+\; \tfrac{1}{2}(1-a'_{n+1})^2 \;>\; b'_n \;+\; \tfrac{1}{2}(1-b'_n)^2.$$

Here it is certainly true that $b_n < b'_n < 1$ when $0 < b_n < 1$; then, however, the extreme right-hand member of the relation is still < 1, and numbers $a_{n+1} < 1$ satisfying it *are available*. There is obviously no obstacle to our making the a_n increase as rapidly as we like towards 1; we can, in particular, have

$$\sum_{n=1}^{\infty} (1-a_n) \;<\; \infty.$$

We henceforth assume that the last precaution has been heeded in the selection of the a_n. The B_n will then lie in their respective *disjoint* rings $a'_n \leqslant |z| \leqslant b'_n$ besides being all included in the cusp-shaped region $|\Im z| \leqslant \frac{1}{2}(1-|z|)^2$ and, of course, in the right half plane. According to what we have already seen, $G(z)$ is equal to the analytic function $b(z)f(z)$ for $z \in \Delta$ *outside* all of the B_n, so then $\partial G(z)/\partial \bar{z} = 0$. *Within* any of the B_n, we have only the estimate $|\partial G(z)/\partial \bar{z}| \leqslant \text{const.}/(1-|z|)^2$.

Because of the configuration of the B_n, $G(z)$ is continuous up to the points of $\partial\Delta \sim \{1\}$. Indeed, when $z \in \Delta$ tends to $e^{i\vartheta} \neq 1$, it must eventually *leave* the region $\{\Re z > 0,\; |\Im z| \leqslant \frac{1}{2}(1-|z|)^2\}$ in which all the B_n lie, and then $G(z)$ becomes equal to $b(z)f(z)$ which has the continuous limit $b(e^{i\vartheta})f(e^{i\vartheta})$ away from 1 on the unit circumference.

If $z \in \Delta$ tends to 1 from *outside* any sector with vertex at 1 of the form $|\arg(1-z)| \leqslant \alpha$, $0 < \alpha < \pi/2$, we have

$$G(z) \longrightarrow 0.$$

To see this, we argue that such z must leave the region $|\Im z| \leqslant \frac{1}{2}(1-|z|)^2$, making $G(z) = b(z)f(z)$. Then, however,

$$\log|b(z)f(z)| \;\leqslant\; -u(z) \;=\; -\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\, d\omega_{\mathscr{D}}(e^{i\vartheta}, z),$$

and it suffices to show that the expression on the right tends to $-\infty$ whenever $z \longrightarrow 1$ from outside any of the sectors just mentioned. This is so due to the fact that $\sum_{n=1}^{\infty} T_n(\vartheta) \longrightarrow \infty$ for $\vartheta \longrightarrow 0$, as may be verified by taking the region

$$\mathscr{E} \;=\; \Delta \sim [1/2,\, 1] \;\subseteq\; \mathscr{D}$$

and comparing harmonic measure for $\mathscr{D}$ with that for $\mathscr{E}$. By the principle of extension of domain, $d\omega_{\mathscr{E}}(e^{i\vartheta}, z) \leqslant d\omega_{\mathscr{D}}(e^{i\vartheta}, z)$ for $z \in \mathscr{E}$, so we need only check that

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\, d\omega_{\mathscr{E}}(e^{i\vartheta}, z) \longrightarrow \infty$$

as $z \longrightarrow 1$ from outside any of the sectors in question. That, however, should be *clear*. Let the reader imagine that $\mathscr{E}$ has been mapped conformally onto the upper half plane so as to take the vertices of its two corners at 1 to -2 and 2, say, and then think about how the *ordinary* Poisson integral corresponding to the last expression must behave as one moves towards -2 or 2 from the upper half plane.

We put finally

$$F(z) = c \exp\left(-K\frac{1+z}{1-z}\right) G(z)$$

for $z \in \Delta$, with c a *small* constant > 0 and K a *large* one. The exponential serves two purposes. It is, in the first place, < 1 in modulus in Δ and continuous up to $\partial\Delta \sim \{1\}$ where it has boundary values of modulus 1. When $z \longrightarrow 1$ from *within* any sector $|\arg(1-z)| \leqslant \alpha$, $0 < \alpha < \pi/2$, the exponential tends to *zero*, making $F(z) \longrightarrow 0$, since $|G(z)| < 1$ in Δ. This, however, is also true when $z \longrightarrow 1$ from *outside* such a sector because then $G(z) \longrightarrow 0$ as we have just seen. Thus,

$$F(z) \longrightarrow 0 \quad \text{for } z \longrightarrow 1,\ z \in \Delta.$$

We have already remarked that $G(z)$ is continuous up to $\partial\Delta \sim \{1\}$, where it coincides with $b(z)f(z)$, so we have

$$F(z) \longrightarrow c\,e^{-Ki\cot(\vartheta/2)}\, b(e^{i\vartheta}) f(e^{i\vartheta})$$

when $z \in \Delta$ tends to $e^{i\vartheta} \neq 1$. Denoting the boundary value on the right by $F(e^{i\vartheta})$, we have $|F(e^{i\vartheta})| = c|f(e^{i\vartheta})| = c\exp(-u(e^{i\vartheta}))$, and this tends to *zero* as $\vartheta \longrightarrow 0$ since $u(e^{i\vartheta}) = \sum_{n=1}^{\infty} T_n(\vartheta)$ then tends to ∞. *The function $F(z)$ thus extends continuously up to the unit circumference* thanks to the factor $\exp(-K(1+z)/(1-z))$. We have $|F(e^{i\vartheta})| = c|f(e^{i\vartheta})| > 0$ for $e^{i\vartheta} \neq 1$, and

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\, d\vartheta = 2\pi \log c + \int_{-\pi}^{\pi} \log|f(e^{i\vartheta})|\, d\vartheta = -\infty.$$

Since $G(z)$ is $\mathscr{C}_\infty$ inside Δ, so is $F(z)$. The second service rendered by the factor $\exp(-K(1+z)/(1-z))$ is to make $\partial F(z)/\partial\bar{z}$ *small* near $\partial\Delta$.

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Outside the B_n, $F(z)$ (like $G(z)$) is *analytic*, so $\partial F(z)/\partial\bar{z} = 0$. *Within* any of the B_n, we use the formula

$$\frac{\partial F(z)}{\partial \bar{z}} = c\exp\left(-K\frac{1+z}{1-z}\right)\frac{\partial G(z)}{\partial \bar{z}},$$

which holds because the exponential is analytic in Δ. The B_n all lie in the right half plane, and in them,

$$|\Im z| \leqslant \tfrac{1}{2}(1-|z|)^2 < \tfrac{1}{2}(1-|z|),$$

whence

$$\Re\frac{1+z}{1-z} \geqslant \frac{\text{const.}}{1-|z|}.$$

This makes

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant c\exp\left(-K\frac{\text{const.}}{1-|z|}\right)\left|\frac{\partial G(z)}{\partial \bar{z}}\right|$$

for z belonging to any of the B_n. As we have seen, the last factor on the right is $\leqslant \text{const.}/(1-|z|)^2$ which, for $|z| < 1$ near 1, is *greatly outweighed* by the exponential. Bearing in mind that $\log(1/|z|) \sim 1-|z|$ for $|z| \longrightarrow 1$, we see that the constants c and K can be adjusted so as to have

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \exp\left(-\frac{1}{\log(1/|z|)}\right),$$

within the B_n at least. But then this holds *outside* them as well (in Δ, including in the neighborhood of 0), because $\partial F(z)/\partial\bar{z} = 0$ there.

$F(z)$ has now been shown to enjoy all the properties enumerated at the beginning of this exposition except the one involving the function $h(\xi)$, not yet constructed. That construction comes almost as an afterthought. Since the sets B_n lie inside the disjoint rings $a'_n \leqslant |z| \leqslant b'_n$, we start by putting $h(\log(1/|z|)) = 1/\log(1/|z|)$ on each of the latter; in view of the preceding relation, this *already implies* that

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \exp(-h(\log(1/|z|)))$$

throughout Δ, no matter *how* $h(\log(1/|z|))$ is defined for the remaining $z \in \Delta$, because the left side is *zero* outside the B_n. To complete the definition of $h(\xi)$ for $0 < \xi < \infty$, we continue to use $h(\log(1/|z|)) = 1/\log(1/|z|)$ on the range $0 < |z| \leqslant a'_1$ and then take $h(\log(1/|z|))$ to be *linear in*

$|z|$ on each of the *complementary* rings

$$b'_n \leqslant |z| \leqslant a'_{n+1}, \qquad n = 1, 2, 3, \ldots .$$

The function $h(\xi)$ we obtain in this fashion is certainly *decreasing* (in ξ); $h(\log(1/|z|))$ is also > 1 for $|z| \geqslant b'_1$, because $b'_1 > a_1 > 2/(2+\sqrt{3}) > 1/e$. $h(\log(1/|z|))$ is moreover $\geqslant 1/\log(1/|z|)$ on the complementary rings, for $1/\log(1/x)$ is a *convex* function of x for $1/e^2 < x < 1$, and $b'_1 > 1/e^2$. In terms of the variable $\xi = \log(1/|z|)$ we therefore have

$$\xi h(\xi) \geqslant 1, \qquad 0 < \xi < \infty.$$

The trick in arranging to have

$$\int_0^1 \log h(\xi)\, d\xi = \infty$$

is to use *linearity* of $h(\log(1/x))$ in x on each interval $b'_n \leqslant x \leqslant a'_{n+1}$ to get lower bounds on the integrals

$$\int_{\log(1/a'_{n+1})}^{\log(1/b'_n)} \log h(\xi)\, d\xi.$$

We have indeed $h(\xi) > 1$ for $\xi \leqslant \log(1/b'_n) \leqslant \log(1/b'_1)$ and $h(\log(1/a'_{n+1})) = 1/\log(1/a'_{n+1})$, so the linearity just mentioned makes $h(\xi) \geqslant 1/2\log(1/a'_{n+1})$ for $(b'_n + a'_{n+1})/2 \leqslant e^{-\xi} \leqslant a'_{n+1}$, i.e., for $\log(1/a'_{n+1}) \leqslant \xi \leqslant \log(2/(b'_n + a'_{n+1}))$. The preceding integral is therefore

$$\geqslant \log\left(\frac{2a'_{n+1}}{a'_{n+1} + b'_n}\right) \cdot \log^+\left(\frac{1}{2\log(1/a'_{n+1})}\right),$$

since $\log h(\xi)$ is > 0 on the whole range of integration. For any *given* value of b'_n, $0 < b'_n < 1$, *the last expression tends to* ∞ *as* $a'_{n+1} \longrightarrow 1$! We can therefore make it $\geqslant 1$ by taking $a'_{n+1} > b'_n$ *close enough to* 1, and that can in turn be achieved by choosing $a_{n+1} = a'_{n+1} + \frac{1}{2}(1 - a'_{n+1})^2$ sufficiently near 1. We therefore *select the successive* a_n *in accordance with this requirement* in carrying out the inductive procedure followed at the beginning of our construction. That will certainly guarantee that $b'_n < a'_{n+1}$ (which we needed), and may obviously be done so as to have $\sum_{n=1}^{\infty}(1 - a_n) < \infty$ (by making the a_n tend more rapidly towards 1 we can only improve matters).

Once the a_n have been specified in this way, we will have

$$\int_{\log(1/a'_{n+1})}^{\log(1/b'_n)} \log h(\xi)\, d\xi \geqslant 1$$

for each n, and therefore

$$\int_0^1 \log h(\xi)\,d\xi \quad = \quad \infty.$$

Our construction of the functions $F(z)$ and $h(\xi)$ with the desired properties is thus complete, and the gap in the second half of article 2 in the addendum to volume I filled in. This means, in particular, that in the hypothesis of Brennan's result (top of p. 574, volume I), *the condition that* $M(v)/v^{1/2}$ *be increasing cannot be replaced by the weaker one that* $M(v)/v^{1/2} \geqslant 2$.

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Errata for volume I

Location	Correction
page 66	At end of the theorem's statement, words in roman should be in italic, and words in italic in roman.
pages 85, 87	In running title, delete bar under second M_n but keep it under first one.
page 102	In heading to §E, delete bar under M_n in first and third $\mathscr{C}_R(\{M_n\})$ but keep it in second one.
page 126, line 8	In statement of theorem, change *determinant* to *determinate*.
page 135, line 11	In displayed formula, change w^* to w^*.
page 136, line 4 from bottom	In displayed formula $\lvert P(x_0)\rvert^2\nu(\{x_0\})$ should stand on the right.
page 177, line 11 from bottom	The sentence beginning 'Since, as we already' should start on a new line, separated by a horizontal space from the preceding one
page 190	In last displayed formula, change x'' to x^n
page 212 and following even numbered pages up to page 232 inclusive	Add to running title: *Comparison of* $\mathscr{C}_W(0)$ *to* $\mathscr{C}_W(0+)$
page 230	In the last two displayed formulas replace $(1-\alpha^2)$ throughout by $\lvert 1-\alpha^2\rvert$.
page 241, line 3	Change b_b^2 in denominator of right-hand expression to b_n^2.
page 270, line 10	Change $F(z)$ to $F(Z)$.
page 287	In figure 69, B_1 and B_2 should designate the lower and upper sides of $\mathscr{D}_0$, not $\mathscr{D}$.

page 379, line 8 from bottom	Change comma after 'theorem' to a full stop, and capitalize 'if'.
page 394, line 3	Change y_1 to y_l.
page 466, last line	Delete full stop.
page 563, line 9	Change 'potential' to 'potentials'.
page 574, line 9 from bottom	Delete full stop after 'following'.
page 604	In running title, '*volume*' should not be capitalized.
page 605	In titles of §§C.1 and C.4 change 'Chapter 8' to 'Chapter VIII'.