IX

Jensen's formula again

The derivations of the two main results in this chapter – Pólya's gap theorem and a *lower* bound for the completeness radius of a set of imaginary exponentials – are both based on the same simple idea: application of Jensen's formula with a circle of varying radius and *moving* centre. I learned about this device from a letter that J.-P. Kahane sent me in 1958 or 1959, where it was used to prove the first of the results just mentioned. Let us begin our discussion with an exposition of that proof.

A. Pólya's gap theorem

Consider a Taylor series expansion

$$f(w) = \sum_0^\infty a_n w^n$$

with radius of convergence equal to 1. The function $f(w)$ must have at least one singularity on the circle $|w| = 1$. It was observed by Hadamard that *if many of the coefficients a_n are zero*, i.e., if, as we say, *the Taylor series has many gaps*, $f(w)$ must have *lots of singularities on the series' circle of convergence*. In a certain sense, *the more gaps the power series has, the more numerous must be the singularities associated thereto on its circle of convergence.*

This phenomenon was studied by Hadamard and by Fabry; the best result was given by Pólya. In order to formulate it, Pólya invented the maximum density bearing his name which has already appeared in Chapter VI.

In this §, it will be convenient to denote by $\mathbb{N}$ *the set of integers* $\geqslant 0$ (and *not just the ones* $\geqslant 1$ as is usually done, and as we will do in §B!). If $\Sigma \subseteq \mathbb{N}$, *we denote by* $n_\Sigma(t)$ *the number of elements of* Σ *in* $[0, t]$, $t \geqslant 0$. The *Pólya* *maximum density* of Σ, studied in §E.3 of Chapter VI, is the quantity

$$D_\Sigma^* = \lim_{\lambda\to 1-}\left(\limsup_{r\to\infty}\frac{n_\Sigma(r)-n_\Sigma(\lambda r)}{(1-\lambda)r}\right).$$

We have shown in the article referred to that the outer limit really does exist for any Σ, and that D_Σ^* is the *minimum of the densities of the measurable sequences containing* Σ. In this §, we use a property of D_Σ^* furnished by the following

Lemma. *Given $\varepsilon > 0$, we have, for $\rho \geqslant \varepsilon r$,*

$$\frac{n_\Sigma(r+\rho)-n_\Sigma(r)}{\rho} \leqslant D_\Sigma^* + \varepsilon$$

when r is large enough (depending on ε).

Proof. According to the above formula, if N is large enough and

$$\lambda = (1+\varepsilon)^{-1/N},$$

we will have

$$\frac{n_\Sigma(r)-n_\Sigma(\lambda r)}{(1-\lambda)r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $r \geqslant R$, say. *Fix* such an N.

When $r \geqslant R$, we certainly have

$$\frac{n_\Sigma(\lambda^{-k-1}r)-n_\Sigma(\lambda^{-k}r)}{(\lambda^{-k-1}-\lambda^{-k})r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $k = 0, 1, 2, \ldots$, so

$$\frac{n_\Sigma(\lambda^{-k}r)-n_\Sigma(r)}{(\lambda^{-k}-1)r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $k = 1, 2, 3, \ldots$. Let

$$\rho \geqslant \varepsilon r = (\lambda^{-N}-1)r.$$

Then, if k is the least integer such that $(\lambda^{-k}-1)r \geqslant \rho$, we have $k \geqslant N$, so, $n_\Sigma(t)$ being increasing,

$$\frac{n_\Sigma(r+\rho)-n_\Sigma(r)}{\rho} \leqslant \frac{n_\Sigma(\lambda^{-k}r)-n_\Sigma(r)}{(\lambda^{-k}-1)r}\cdot\frac{(\lambda^{-k}-1)r}{\rho}$$

$$< \left(D_\Sigma^* + \frac{\varepsilon}{2}\right)\frac{\lambda^{-k}-1}{\lambda^{-k+1}-1}$$

$$\leqslant \left(D_\Sigma^* + \frac{\varepsilon}{2}\right)\frac{\lambda^{-N}-1}{\lambda^{-N+1}-1} \quad = \quad \frac{\varepsilon}{(1+\varepsilon)^{(N-1)/N} - 1}\left(D_\Sigma^* + \frac{\varepsilon}{2}\right)$$

when $r \geqslant R$. If N is chosen large enough to begin with, the last number is $\leqslant D_\Sigma^* + \varepsilon$. This does it.

Theorem (Pólya). *Let the power series*

$$f(w) = \sum_{n\in\Sigma} a_n w^n$$

have radius of convergence 1. *Then, on every arc of* $\{|w| = 1\}$ *with length* $> 2\pi D_\Sigma^*$, $f(w)$ *has at least one singularity.*

Proof (Kahane). Assume that $f(w)$ can be continued analytically through an arc on the unit circle of length $> 2\pi D$, which we may wlog take to be symmetric about -1. We then have to prove that $D \leqslant D_\Sigma^*$. We may of course take $D > 0$. There is also no loss of generality in assuming $D < 1$, for here the power series' circle of convergence, which does include at least one singularity of $f(w)$, has length 2π.

Pick any $\delta > 0$. In the formula

$$a_n = \frac{1}{2\pi i}\int_{|w|=e^{-\delta}} f(w) w^{-n-1}\,dw$$

(we are, of course, taking a_n as *zero* for $n \notin \Sigma$, $n \geqslant 0$) one may, thanks to the analyticity of $f(w)$, *deform* the path of integration $\{|w| = e^{-\delta}\}$ to the contour Γ_δ shown here:

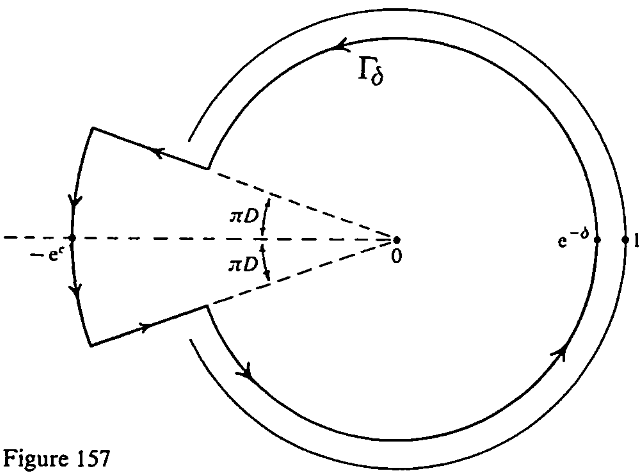

Figure 157

The quantity $c > 0$ is fixed once D is given, and independent of δ.

In the integral around Γ_δ, make the change of variable $w = e^{-s}$, where $s = \sigma + i\tau$ with τ ranging from $-\pi$ to π. Our expression then goes over into

$$\frac{1}{2\pi i}\int_{\gamma_\delta} f(e^{-s})e^{ns}\,ds \quad = \quad a_n$$

with this path γ_δ:

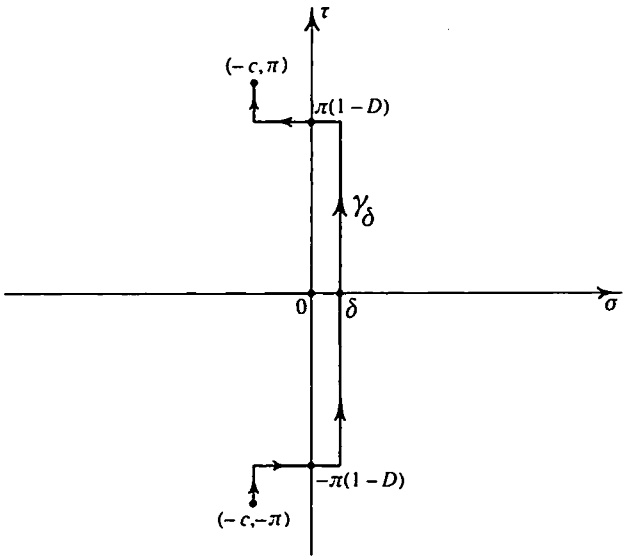

Figure 158

Write

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_{\gamma_\delta} f(e^{-s})e^{zs}\,ds$$

so that $F(n) = a_n$ for $n \in \mathbb{N}$ (and is hence *zero* for $n \in \mathbb{N} \sim \Sigma$); $F(z)$ is of course entire and of exponential type. We break up the integral along γ_δ into three pieces, I, II and III, coming from the *front vertical*, *horizontal* and *rear vertical* parts of γ_δ respectively.

On the *front vertical* part of γ_δ, $|f(e^{-s})| \leqslant M_\delta$ and $|e^{sz}| \leqslant e^{\delta x + \pi(1-D)|y|}$ (writing as usual $z = x + iy$); hence

$$|\mathrm{I}| \leqslant M_\delta e^{\delta x + \pi(1-D)|y|}$$

On the *horizontal* parts of γ_δ, $|f(e^{-s})| \leqslant C$ (a number independent of δ, by the way), and $|e^{sz}| \leqslant e^{\delta x + \pi(1-D)|y|}$ for $x > 0$, whence

$$|\mathrm{II}| \leqslant Ce^{\delta x + \pi(1-D)|y|}, \quad x > 0.$$

Finally, on the *rear vertical* parts of γ_δ, $|f(e^{-s})| \leqslant C$ and $|e^{sz}| \leqslant e^{-cx+\pi|y|}$ for $x > 0$, making

$$|\mathrm{III}| \leqslant Ce^{-cx+\pi|y|}, \quad x > 0.$$

Adding these three estimates, we get

$$|F(z)| \leqslant (M_\delta + C)e^{\delta x + \pi(1-D)|y|} + Ce^{-cx+\pi|y|}$$

for $x > 0$. Since $c > 0$, the *second* term on the right will be $\leqslant$ the *first* in the sector

$$S = \left\{ z: \; |\Im z| \leqslant \frac{c}{\pi D} \Re z \right\}$$

with opening *independent of* δ. We thus have

$$|F(z)| \leqslant K_\delta e^{\delta x + \pi(1-D)|y|} \quad \text{for } z \in S,$$

K_δ being a constant depending on δ. The idea here is that *the availability, for* $f(w)$, *of an analytic continuation through the arc* $\{e^{i\vartheta}: |\vartheta - \pi| \leqslant \pi D\}$ *has made it possible for us to diminish the term* $\pi|y|$, *which would normally occur in the exponent on the right, to* $\pi(1-D)|y|$, thanks to the term $-cx$ figuring in the previous expression.

Because $\sum_{n\in\Sigma} a_n w^n$ has *radius of convergence* 1, there is a subsequence Σ' of Σ with

$$\frac{\log|a_n|}{n} \longrightarrow 0 \quad \text{for } n \longrightarrow \infty \text{ in } \Sigma'.$$

Let 2α be the opening (*independent* of δ) of our sector S, i.e.,

$$\alpha = \arctan \frac{c}{\pi D}$$

With $n \in \Sigma'$, write Jensen's formula for $F(z)$ and the circle of radius $n \sin\alpha$ about n (this is Kahane's idea). That is just

$$\int_0^{n\sin\alpha} \frac{N(\rho, n)}{\rho} d\rho = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(n + n\sin\alpha\, e^{i\vartheta})|d\vartheta - \log|a_n|,$$

where $N(\rho, n)$ denotes the number of zeros of $F(z)$ in the disk $\{|z - n| \leqslant \rho\}$.

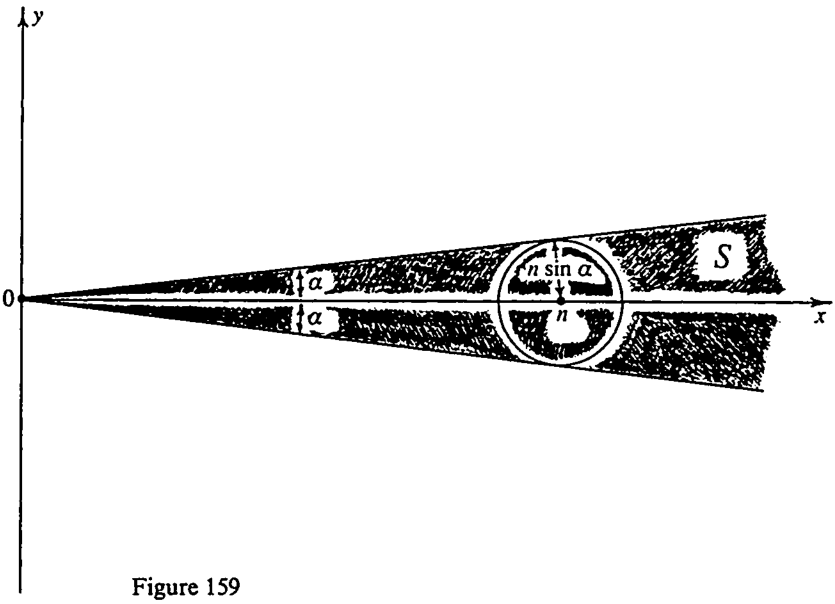

Figure 159

By the above estimate on $|F(z)|$ for $z \in S$, the *right side* of the relation just written is

$$\leqslant \quad \log K_\delta \;+\; \frac{1}{2\pi}\int_{-\pi}^{\pi} (\delta n + \delta n \sin\alpha \cos\vartheta + n\sin\alpha\cdot\pi(1-D)|\sin\vartheta|)\,d\vartheta \\ -\; \log|a_n|$$

$$= \quad \log K_\delta + \delta n + 2(1-D)n\sin\alpha - \log|a_n|.$$

The *left side* we estimate *from below*, using the lemma. Since $F(m) = a_m = 0$ for $m \in \mathbb{N} \sim \Sigma$, we have, for $0 < \rho \leqslant n$,

$$N(\rho, n) \;\geqslant\; \text{number of integers in } [n-\rho,\ n+\rho] \;-\; \text{number of elements of } \Sigma \text{ in } [n-\rho,\ n+\rho]$$
$$\geqslant\; 2\rho \;-\; (n_\Sigma(n+\rho) - n_\Sigma(n-\rho)) \;-\; 2.$$

Fix any ε, $0 < \varepsilon < \sin\alpha$. According to the lemma, for n sufficiently large,

$$n_\Sigma(n+\rho) - n_\Sigma(n-\rho) \;\leqslant\; 2\rho(D_\Sigma^* + \varepsilon)$$

when $\varepsilon(1-\sin\alpha)n \leqslant 2\rho \leqslant 2n\sin\alpha$, so, for such ρ (and large n),

$$N(\rho, n) \;\geqslant\; 2(1 - D_\Sigma^* - \varepsilon)\rho \;-\; 2.$$

Hence, since $n_\Sigma(t)$ increases,

$$\int_0^{n\sin\alpha} \frac{N(\rho, n)}{\rho}\,\mathrm{d}\rho \;\geqslant\; 2(1 - D^*_\Sigma - \varepsilon)\left(\sin\alpha - \frac{\varepsilon(1-\sin\alpha)}{2}\right)n$$

$$-\; 2\log\frac{2\sin\alpha}{\varepsilon(1-\sin\alpha)}.$$

Use this inequality together with the preceding estimate for the right side of the above Jensen formula. After dividing by $2n\sin\alpha$, one finds that

$$(1 - D^*_\Sigma - \varepsilon)\left(1 - \frac{\varepsilon(1-\sin\alpha)}{2\sin\alpha}\right) \;\leqslant\; \frac{\delta}{2\sin\alpha} + 1 - D$$

$$-\; \frac{\log|a_n|}{2n\sin\alpha} + O\left(\frac{1}{n}\right)$$

for large n, whence, making $n \longrightarrow \infty$ in Σ',

$$(1 - D^*_\Sigma - \varepsilon)\left(1 - \frac{\varepsilon(1-\sin\alpha)}{2\sin\alpha}\right) \;\leqslant\; 1 - D + \frac{\delta}{2\sin\alpha},$$

on account of the behaviour of $\log|a_n|$ for $n \in \Sigma'$.

The quantity ε, $0 < \varepsilon < \sin\alpha$, is arbitrary, and so is $\delta > 0$ with, as we have remarked, the opening 2α of S independent of δ. We thence deduce from the previous relation that $1 - D^*_\Sigma \leqslant 1 - D$, i.e., that

$$D \;\leqslant\; D^*_\Sigma.$$

This, however, is what we had to prove. We are done.

Remark. We see from the proof that it is really the presence in the Taylor series of *many gaps 'near' those $n \in \Sigma$ for which $|a_n|$ is 'big'* (the $n \in \Sigma'$) that gives rise to *large numbers of singularites* on the circle of convergence. The reader is invited to formulate a precise statement of this observation, obtaining a theorem in which the behaviour of the a_n and that of Σ both figure.

Pólya's gap theorem has various generalizations to Dirichlet series. For these, the reader should first look in the last chapter of Boas' book, after which the one by Levinson may be consulted. The most useful work on this subject is, however, the somewhat older one of V. Bernstein. Two of Mandelbrojt's books – the one published in 1952 and an earlier Rice Institute pamphlet on Dirichlet series – also contain interesting material, as does J.-P. Kahane's thesis, beginning with part II. There is, in addition, a recent monograph by Leontiev.

B. Scholium. A converse to Pólya's gap theorem

The quantity D_Σ^* figuring in the result of the preceding § is a kind of *upper density* for sequences Σ of positive integers. Before continuing with the main material of this chapter, it is natural to ask whether D_Σ^* is *the right kind of density measure to use* for a sequence Σ when investigating the distribution of the singularities associated with

$$\sum_{n\in\Sigma} a_n w^n$$

on that series' circle of convergence. Maybe there is always a singularity on each arc of that circle having opening greater than $2\pi d_\Sigma$, with d_Σ a quantity $\leqslant D_\Sigma^*$ associated to Σ which is really $< D_\Sigma^*$ for some sequences Σ. It turns out that *this is not the case*; D_Σ^* is *always* the critical parameter associated with the sequence Σ insofar as distribution of singularities on the circle of convergence is concerned.

This fact, which shows Pólya's gap theorem to be *definitive*, is not well known in spite of its clear scientific importance. It is the content of the following

Converse to Polya's gap theorem *Given any sequence Σ of positive integers with Pólya maximum density $D_\Sigma^* > 0$, there is, for any δ, $0 < \delta < D_\Sigma^*$, a Taylor series*

$$\sum_{n\in\Sigma} a_n w^n$$

with radius of convergence 1, *equal, for $|w| < 1$, to a function which can be continued analytically through the arc*

$$\{e^{i\vartheta}:\ |\vartheta| < \pi(D_\Sigma^* - \delta)\}.$$

The present § is devoted to the establishment of this result in its full generality.

1. Special case. Σ measurable and of density $D > 0$.

If $\lim_{t\to\infty} n_\Sigma(t)/t$ exists and equals a number $D > 0$ ($n_\Sigma(t)$ denoting the number of elements of Σ in $[0,\ t]$), the converse* to Pólya's theorem is easy – I think it is due to Pólya himself. The contour integration technique used to study this case goes back to Lindelöf; it was extensive-

* in a strengthened version, with analytic continuation through the arc $|\vartheta| < \pi D_\Sigma^* = \pi D$

ly used by V. Bernstein in his work on Dirichlet series, and later on by L. Schwartz in his thesis on sums of exponentials.

Restricting our attention to sequences Σ of strictly positive integers clearly involves no loss in generality; we do so throughout the present § because that makes certain formulas somewhat simpler. Denote by $\mathbb{N}$ *the set of integers* > 0 (N.B. *this is different from the notation of* §A, *where* $\mathbb{N}$ *also included* 0), and by Λ the sequence of *positive integers complementary to* Σ, i.e.,

$$\Lambda = \mathbb{N} \sim \Sigma.$$

For $t \geqslant 0$, we simply write $n(t)$ for the *number of elements of* Λ (N.B.!) in $[0, t]$. Put*

$$C(z) = \prod_{n \in \Lambda} \left(1 - \frac{z^2}{n^2}\right);$$

in the present situation

$$\frac{n(t)}{t} \longrightarrow 1 - D \quad \text{for} \quad t \longrightarrow \infty$$

and on account of this, $C(z)$ turns out to be an entire function of exponential type with quite regular behaviour.

Problem 29

(a) By writing $|\log C(z)|$ as a Stieltjes integral and integrating by parts, show that

$$\frac{\log|C(iy)|}{|y|} \longrightarrow \pi(1 - D)$$

for $y \longrightarrow \pm\infty$

(b) Show that for $x > 0$,

$$\log|C(x)| = 2\int_0^1 \left(\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)\right)\frac{d\tau}{1 - \tau^2}.$$

(Hint: First write the left side as a Stieltjes integral, then integrate by parts. Make appropriate changes of variable in the resulting expression.)

(c) Hence show that for $x > 0$,

$$\log|C(x)| \leqslant 2n(x)\log\frac{1}{\gamma} + 2\int_0^\gamma \left(\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)\right)\frac{d\tau}{1 - \tau^2},$$

with γ any number between 0 and 1.

* When $D = 1$, the complementary sequence Λ has density zero and may even be empty. In the last circumstance we take $C(z) \equiv 1$; the function $f(w)$ figuring in the construction given below then reduces simply to $w/\pi(1 + w)$.

(d) By making an appropriate choice of the number γ in (c), show that $\log|C(x)| \leqslant \varepsilon x$ for large enough x, $\varepsilon > 0$ being arbitrary.

(e) Use an appropriate Phragmén–Lindelöf argument to deduce from (a) and (d) that

$$\limsup_{r\to\infty} \frac{\log|C(re^{i\vartheta})|}{r} \leqslant \pi(1-D)|\sin\vartheta|.$$

(f) Show that in fact

$$\frac{\log|C(n)|}{n} \longrightarrow 0 \quad \text{for} \quad n \longrightarrow \infty \text{ in } \Sigma,$$

and that we have *equality* in the result of (e).

(Hint: Form the function

$$K(z) = \prod_{n\in\Sigma}\left(1-\frac{z^2}{n^2}\right);$$

then, as in (e),

$$\limsup_{r\to\infty} \frac{\log|K(re^{i\vartheta})|}{r} \leqslant \pi D|\sin\vartheta|.$$

Show that the same result holds if $K(re^{i\vartheta})$ is replaced by $K'(re^{i\vartheta})$. Observe that

$$\pi z K(z) C(z) = \sin \pi z.$$

Look at the derivative of the left-hand side at points $n\in\Sigma$.)

We are going to use the function $C(z)$ to construct a power series

$$\sum_{n\in\Sigma} a_n w^n$$

having *radius of convergence* 1, and representing a function which can be *analytically continued into the whole sector* $|\arg w| < \pi D$.

Start by putting

$$f(w) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{C(\zeta)}{\sin\pi\zeta} w^{\zeta}\, d\zeta$$

for $|\arg w| < \pi D$. Given any $\varepsilon > 0$, we see, by part (e) of the above problem, that

$$\left|\frac{C(\frac{1}{2}+i\eta)}{\sin\pi(\frac{1}{2}+i\eta)}\right| \leqslant \frac{\text{const.}}{\cosh\pi\eta} e^{(\pi(1-D)+\varepsilon)|\eta|}$$