COMPLEX FUNCTIONS
an algebraic and geometric viewpoint

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Introduction

Throughout the nineteenth century, the attention of the mathematical world was, to a large extent, concentrated on complex function theory, that is, the study of meromorphic functions of a complex variable. Some of the greatest mathematicians of that period, including Gauss, Cauchy, Abel, Jacobi, Eisenstein, Riemann, Weierstrass, Klein and Poincaré, made substantial contributions to this theory, and their work (mainly on what we would now regard as specific, concrete problems) led to the subsequent development of more general and abstract theories throughout pure mathematics in the present century. Because of its central position, directly linked with analysis, algebra, number theory, potential theory, geometry and topology, complex function theory makes an interesting and important topic for study, especially at undergraduate level: it has a good balance between general theory and particular examples, it illustrates the development of mathematical thought, and it encourages the student to think of mathematics as a unified subject rather than (as it is often taught) as a collection of mutually disjoint topics.

Even though the subject matter of this book is classical, it has recently assumed great importance in several different areas of mathematics. For example, the recent work on W. Thurston on 3-manifolds shows the vital importance of hyperbolic geometry and Möbius transformations to this rapidly developing subject; a totally different example is given by the work of J.G. Thompson, J.H. Conway and others on the 'monster' simple group, where the J-function, studied in Chapter 6, seems to play an important (and, at the time of writing, rather mysterious) role. Thus many active mathematicians, whose work may not involve classical complex function theory directly, will nevertheless need to become familiar with certain aspects of the theory, and we hope that they find our elementary approach of use, at least initially.

This book is based on a final-year undergraduate course at the University of Southampton, taught first by D.S. and then by G.A.J., though we have also included some additional material, generally at the end of a chapter, suitable for graduates or for more advanced undergraduates. Our aim, both in the lecture-course and in this book, is to teach some of the main ideas about complex functions and Riemann surfaces, assuming only the basic algebraic, analytic and topological theories covered by students in their first and second years at university, and to show how these three subjects can be combined to throw light on a single, specific topic. (Of course, this involves reversing the historical development of the subject: to the modern mind, general theories often appear more elementary and accessible than the particular examples from which they grew.) Shortages of space and time forced us to ignore the connections with, for example, number theory and potential theory, interesting though they are; in any case, there are excellent books on these topics.

In Chapter 1 we use stereographic projection to show how the addition of a single point ∞ to $\mathbb{C}$ transforms the plane into a sphere, the Riemann sphere $\Sigma = \mathbb{C} \cup \{\infty\}$, and we describe the meromorphic functions $f:\Sigma \to \Sigma$ from both an algebraic and a topological point of view. The main result, which is a typical connection between analytic and algebraic concepts, is that $f:\Sigma \to \Sigma$ is meromorphic if and only if it is a rational function.

Chapter 2 concerns the automorphisms of Σ, that is, the meromorphic bijections $f:\Sigma \to \Sigma$, or equivalently the Möbius transformations

$$f(z) = \frac{az + b}{cz + d}, \qquad (*)$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. These transformations form a group $\operatorname{Aut}\Sigma$ under composition, and the emphasis of this chapter is mainly group-theoretic; for example, the finite subgroups of $\operatorname{Aut}\Sigma$ are determined, and the cross-ratio λ is introduced in order to study the transitivity properties of $\operatorname{Aut}\Sigma$. We also consider some of the geometric properties of Möbius transformations (especially their relationship with circles in Σ), and the way in which $\operatorname{Aut}\Sigma$ acts as the Galois group of the field of all meromorphic functions on Σ.

In Chapter 3 we study periodic meromorphic functions on $\mathbb{C}$; these fall into two classes, the simply and doubly periodic, according to whether the group of periods has one or two generators. After briefly considering simply periodic functions (such as the exponential and trigonometric functions), and their Fourier series expansions, we devote the rest of the chapter to doubly periodic functions, called elliptic functions because they first arose from attempts to evaluate certain integrals associated with the formula for the circumference of an ellipse. The periods of such a function form a lattice, that is, a subgroup of $\mathbb{C}$ (under addition) generated by two complex numbers which are linearly independent over $\mathbb{R}$. Just as the rational functions are the meromorphic functions on the sphere Σ, the elliptic functions can be regarded as the meromorphic functions on the torus $\mathbb{C}/\Omega$ whose elements are the cosets in $\mathbb{C}$ of a lattice Ω. There are many close analogies between rational and elliptic functions, mainly based on the fact that both Σ and $\mathbb{C}/\Omega$ are compact surfaces: for example, an important consequence of Liouville's theorem is that an analytic function on either of these surfaces must be constant. However, in the case of the torus (as opposed to the sphere) the construction of non-constant meromorphic functions represents a substantial problem: by imitating the infinite product expansion of the simply periodic function $\sin(z)$, we introduce the Weierstrass function $\sigma(z)$, and then by successive differentiation we obtain the Weierstrass functions $\zeta(z)$ and $\wp(z)$, the last of these being elliptic and not constant. This approach is an alternative to the now-traditional direct construction of $\wp$ (outlined in the exercises) by infinite series, and it involves some elementary properties of uniform and normal convergence of infinite series and products; these properties, important in their own right, are outlined in §3.7 and §3.8. The rest of this chapter is concerned with deriving properties of the functions $\wp$, ζ and σ, and hence of all elliptic functions. For example, the elliptic functions are precisely the rational functions of $\wp$ and its derivative $\wp'$, these two functions being related by an ordinary differential equation $\wp' = \sqrt{(p(\wp))}$, where p is a cubic polynomial; the functions ζ and σ, though not themselves elliptic, are important for the construction of elliptic functions with certain properties such as specific zeros, poles or principal parts. The chapter closes with the addition theorem, expressing $\wp(z_1 + z_2)$ in terms of $\wp(z_1)$ and $\wp(z_2)$; historically this should come first, since it was the work of Fagnano and Euler on addition theorems for elliptic integrals which eventually led to the discovery of elliptic functions.

Whereas Chapters 1–3 can be regarded as concerned with meromorphic functions on two specific surfaces Σ and $\mathbb{C}/\Omega$, the theme of Chapter 4 is to take a function f (possibly many-valued, such as $\log(z)$) and to find the most natural surface to regard as its domain of definition. More precisely, we replace f by a single-valued function ϕ which represents the different branches of f; the domain of ϕ, chosen to be as large as possible subject to ϕ representing f locally, is called the Riemann surface S of f. The construction of ϕ and S involves the concepts of analytic and meromorphic continuation, together with the monodromy theorem which allow us to construct single-valued functions on simply connected regions; several examples, such as $\log(z)$ and $\sqrt{p(z)}$ (p a polynomial) are studied in detail. In the second half of the chapter we consider Riemann surfaces as abstract topological objects in their own right, not necessarily obtained from functions. By introducing the concept of the germ of a meromorphic function we show that every algebraic function determines a compact Riemann surface, and we prove the Riemann–Hurwitz formula for the genus of such a surface. Every Riemann surface is conformally equivalent (that is, isomorphic) to a quotient surface $\hat{S}/G$, where $\hat{S}$ (the universal covering space of S) is a simply connected Riemann surface and G is a discrete group of automorphisms of $\hat{S}$; for example, a torus S has the form $\mathbb{C}/\Omega$ for some lattice Ω which acts as a discrete group of translations of $\hat{S} = \mathbb{C}$. By the uniformisation theorem of Poincaré and Koebe (the proof of which is beyond the scope of this book), $\hat{S}$ is conformally equivalent to $\mathbb{C}$, Σ or $\mathscr{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, so we conclude the chapter by determining the automorphism groups of these three important surfaces.

With just a few exceptions, most Riemann surfaces S have as their universal covering space $\hat{S}$ the upper half-plane $\mathscr{U}$, and Chapter 5 is devoted to the study of this particular surface and its discrete groups of automorphisms. These are the Fuchsian groups, consisting of Möbius transformations (*) with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$; by defining an appropriate metric on $\mathscr{U}$ (the hyperbolic metric) we can regard $\mathscr{U}$ as a model of the hyperbolic plane, with these transformations acting as isometries. This situation is similar to, but considerably more complicated than earlier cases where we considered automorphisms of Σ and of $\mathbb{C}$. Using hyperbolic geometry we study Fuchsian groups G, the associated quotient surfaces $S = \mathscr{U}/G$, and their automorphism groups $\mathrm{Aut}\, S$. For example, if S is compact and has genus $g > 1$, then $|\mathrm{Aut}\, S| \leqslant 84(g - 1)$, and we shall give an algebraic description of the Fuchsian groups G and the groups $\mathrm{Aut}\, S$ (the Hurwitz groups) for which this bound is attained.

Chapter 6 concerns perhaps the most important of all Fuchsian groups, the modular group Γ consisting of the Möbius transformations (*) with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. This group and its action on $\mathscr{U}$ arise from the problem of determining all Riemann surfaces of genus 1, or equivalently, all similarity classes of lattices $\Omega \subset \mathbb{C}$; there is one conformal equivalence class of such surfaces for each orbit of Γ on $\mathscr{U}$. For example, if $p(z)$ is a cubic polynomial with distinct roots then the Riemann surface S of $\sqrt{p(z)}$ has genus 1, and we shall show that S is conformally equivalent to a torus $\mathbb{C}/\Omega$ by finding a lattice Ω for which the associated Weierstrass elliptic function $\wp$ satisfies the differential equation $\wp' = \sqrt{p(\wp)}$; this is done by constructing an analytic function $J: \mathscr{U} \to \mathbb{C}$, invariant under the action of Γ on $\mathscr{U}$, and using J to select the orbit of Γ on $\mathscr{U}$ corresponding to the appropriate lattice

Ω. (This function J is closely associated with the cross-ratio function λ introduced in Chapter 2.) From its action on $\mathscr{U}$ we obtain generators and relations for Γ, and hence we are able to consider its homomorphic images, many of which (such as the Hurwitz groups) have already appeared in earlier chapters. Finally we consider the quotient surfaces of $\mathscr{U}$ corresponding to normal subgroups of Γ, including the congruence subgroups obtained by mapping the coefficients a, b, c, d in (*) into the ring of integers mod (n), for positive integers n.

The Appendix contains statements of the main elementary results we have assumed about complex functions, and also some of the basic facts (less well known than they should be) about polynomials and their discriminants.

Clearly, this book contains considerably more material than could possibly be taught in the 36-lecture course on which it is based: a typical course would cover Chapter 1 and about half each of Chapters 2, 3 and 4. In fact, since the chapters are fairly self-contained, this book could be used as the basis for more specialised courses on several different subjects, such as the Riemann sphere and its Möbius transformations (Chapters 1 and 2), elliptic functions (Chapters 1 and 3), analytic continuation and Riemann surfaces (Chapters 1 and 4), and hyperbolic geometry (Chapter 5 and parts of Chapter 4), while for more advanced students Chapter 6 would serve as an introduction to the modular group, leading on to the more detailed treatments in the books by Rankin and Schoeneberg.

In writing a book of this nature, one acquires many debts of gratitude. Our first is to the great men, named above, who founded this subject; the ideas in this book are all theirs, and our only contribution has been to become sufficiently enthusiastic to wish to teach, and then to write down, what they did. One learns mathematics and how to communicate it from many sources and people, far too numerous to mention here; let us simply say that without Murray Macbeath and Peter Neumann we could never have written this book. Alan Beardon, who read the early drafts of the manuscript, saved us from a number of embarrassing solecisms and ambiguities with his detailed criticisms and generous advice, while Robin Bryant, John Thornton and Mary Tyrer-Jones also gave us invaluable help by checking some of the later drafts and the exercises; any remaining blemishes are entirely of our own making. Beryl Betts, June Kerry and Marie Turner deserve our heartfelt thanks for transforming our handwritten scrawls into presentable typescript, and similarly Rose Cassell for her careful drawing of the diagrams; we are also grateful to the staff of the

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Numbering of theorems

Theorems are numbered according to their chapter and section. For example, Theorem 5.7.2 is in Chapter 5, Section 7. Equations are numbered in the same way. The only exceptions are the theorems in the appendix, which are numbered Theorem A.1, Theorem A.2, etc.