The One-Dimensional Heat Equation

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The One-Dimensional Heat Equation

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To my family,
Joyce,
Carolyn, Sue, and Paul

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the nonspecialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

Foreword

The one-dimensional heat equation, first studied by Fourier at the beginning of the 19th century in his celebrated volume on the analytical theory of heat, has become during the intervening century and a half the paradigm for the very extensive study of parabolic partial differential equations, linear and nonlinear. The present volume is a systematic development of a variety of aspects of this paradigm, of which many have not yet received an extension to the multidimensional space-variable case. Of particular interest are the discussions of free-boundary-value problems such as the one-phase Stefan problem, inverse problems, and some classes of not-well-posed problems.

This type of treatment using concrete analytic machinery for the detailed study of this very familiar and widely applicable partial differential equation should prove valuable as a textbook for courses that try to present basic aspects of partial differential equations in simple but useful cases (like the heat equation in one dimension), where the basic concepts are relatively unobscured by the technical problems and complications encountered in the more general classes of equations. The treatment is reasonably complete and can be followed by scientists who do not necessarily have the mathematical experience necessary for some of the more elaborate treatises on general parabolic equations. In addition, the relative completeness of the presentation for this case makes the volume suitable as a reference book for specific results in this area, for which a reference by specialization of more general results is inappropriate. As the author remarks, a variety of analytical techniques are brought to bear under reasonably simple hypotheses, thus illuminating the relative power and effect of the different classes of methods (and, in particular, the contrast between constructive methods and the use of *a priori* bounds).

In summary, the volume is a useful contribution to the effort to bring the material of the research literature in analysis into a form useful to applied mathematicians and mathematically oriented specialists in the sciences.

FELIX E. BROWDER

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Frontmatter
More information

Preface

For more than two decades, part of my research has been directed at various questions involving the heat equation. In this volume I have interwoven much of my research and the research of others with the classical material, at a presentation level suitable for upper-division and beginning graduate mathematics, engineering, and science students. However, I have also intentionally written the material as a monograph and/or information source book. After the first six chapters of standard classical material, each chapter is written as a self-contained unit, except for an occasional reference to elementary definitions, theorems, and lemmas in previous chapters. Consequently, I believe that material can be drawn from the book as needed for a variety of courses, such as a standard course in partial differential equations, a course in initial-boundary-value problems for the heat equation, a course in not-well-posed problems and their numerical solution, a course in free-boundary-value problems, a course in parameter identification, and several others.

The treatment begins with a chapter of preliminary material in order to reduce the need for other reference material. This is followed by six chapters containing the standard basic material for the heat equation, such as the weak maximum principle, elementary solutions, and fundamental solution, and the usual initial- and/or boundary-value problems. One exception to the usual material is the treatment of the noncharacteristic Cauchy problem in Chapter 2. Utilizing the solution representations in Chapters 3 through 6, a fair number of initial-boundary-value problems are reduced to an equivalent system of integral equations in Chapter 7. A relevant treatment of integral equations is presented in Chapter 8. Chapter 9 deals with time-periodic problems. The property of analyticity of solutions is dealt with in Chapter 10, and estimation machinery is developed for utilization in the estimation of continuous dependence of solutions of some not-well-posed problems presented in Chapter 11. Chapter 12 discusses some numerical methods for not-well-posed problems. Chapter 13 contains a treatment of the inverse problem of the determination of an unknown coefficient from overspecified boundary data. Chapter 14 begins the discussion of moving-boundary problems and introduces a stiffening of the level of presentation through Chapter 16. Chapter 15 contains some general properties of solutions that are useful in the discussion of the general initial-boundary-value problem via the Perron–Poincaré method in Chapter 16. The results in Chapter 15 also find application in the discussion of the one-phase Stefan problem in Chapters 17 and 18. Chapter 19 contains a discussion of the inhomogeneous heat equation. Chapter 20 is an application of Chapter 19 to some special quasilinear parabolic equations. Finally, the volume concludes with a bibliography to the literature on the heat equation. The bibliography is divided by time periods, and each of the more recent time periods is subdivided by topics.

The study of partial differential equations has been in progress for more than 200 years and the object of this study, $u_t = u_{xx}$, for more than 150 years. Partial differential equations is a branch of analysis which, including the calculus, has been under development for more than 300 years. Almost every branch of analysis can be and has been brought to bear upon the study of solutions of partial differential equations.

I have assumed that the reader has had a typical undergraduate science and engineering mathematics sequence, which usually attempts to survey calculus, vector analysis, Fourier series and transforms, complex analysis, ordinary and partial differential equations. I have made no attempt to redo a reader's mathematical education. Also, I make no apology for the appearance of "rigor" or "lack of rigor" in any discussion throughout the text.

I am happy to acknowledge the assistance and/or support of the following:

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