Chapter 0

Preliminaries

0.0. SOME INEQUALITIES

We begin with the fact that $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$, from whence follows

$$2ab \leq a^2 + b^2 \tag{0.0.1}$$

for any real a and b. Dividing by 2 and replacing a by $\sqrt{2\varepsilon}\, a$ and b by $b/\sqrt{2\varepsilon}$, we obtain

$$ab \leq \varepsilon a^2 + (4\varepsilon)^{-1} b^2 \tag{0.0.2}$$

for any real a, b and $\varepsilon > 0$.

From the inequality

$$0 \leq \sum (a_i\lambda + b_i)^2 = \left(\sum a_i^2\right)\lambda^2 + 2\left(\sum a_i b_i\right)\lambda + \left(\sum b_i^2\right), \tag{0.0.3}$$

it follows that the discriminant of the quadratic in λ is nonnegative. This yields Schwarz's inequality:

$$\left|\sum a_i b_i\right| \leq \left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2}. \tag{0.0.4}$$

The proof generalizes easily to definite integrals:

$$\left|\int fg\, dx\right| \leq \left(\int f^2\, dx\right)^{1/2}\left(\int g^2\, dx\right)^{1/2}. \tag{0.0.5}$$

The calculus can be employed to show that for nonnegative a, b, and $\alpha, 0 < \alpha < 1$,

$$(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}. \tag{0.0.6}$$

Clearly, inequality (0.0.6) is valid for a or b or both equal to zero. So we can assume that $0 < a < b$. Using the Mean-Value Theorem,

$$(a+b)^{\alpha} = b^{\alpha}\left(1+\frac{a}{b}\right)^{\alpha} = b^{\alpha}\left\{1+\alpha(1+\xi)^{\alpha-1}\left(\frac{a}{b}\right)\right\}$$

$$\leq b^{\alpha}\left\{1+\alpha\left(\frac{a}{b}\right)^{\alpha-1}\left(\frac{a}{b}\right)\right\} = \alpha a^{\alpha} + b^{\alpha} < a^{\alpha} + b^{\alpha}.$$

By induction it follows for $a_i \geq 0$ and $0 < \alpha < 1$, that

$$\left(\sum a_i\right)^{\alpha} \leq \sum a_i^{\alpha}. \tag{0.0.7}$$

We will need the inequality

$$(a+b)^{-2\beta} \leq a^{-\beta} b^{-\beta}, \tag{0.0.8}$$

where $0 < \beta < 1$ and $0 < a, b$. This follows from $(a+b)^2 > ab$ since x^{β} is monotone increasing.

We also recall the elementary estimates

$$\exp\{-x\} \leq p! x^{-p} \tag{0.0.9}$$

for any nonnegative integer p, and for $x > 0$,

$$x^{\alpha}\exp\{-\beta x\} \leq \left(\frac{\alpha}{\beta}\right)^{\alpha}\exp\{-\alpha\} \leq \left(\frac{\alpha}{\beta}\right)^{\alpha} \tag{0.0.10}$$

for any positive α and β. From (0.0.10) it follows that

$$x^{\alpha}\exp\{-\gamma x\} \leq C\exp\{-\beta x\} \tag{0.0.11}$$

for some positive C depending on the positive α, γ, and β, provided that $0 < \beta < \gamma$.

0.1. SEQUENCES AND SERIES OF CONTINUOUS FUNCTIONS, THE ASCOLI–ARZELA THEOREM, AND THE WEIERSTRASS APPROXIMATION THEOREM

The set of functions $f_n = f_n(x)$, $n = 1, 2, \ldots$, which are defined over a common domain, is called a *sequence* of *functions*. If for each $x \in D$, the common domain of definition,

$$\lim_{n\to\infty} f_n(x) = f(x), \tag{0.1.1}$$

then we say that the sequence *converges pointwise* to the function f. Note that buried in the symbolic statement is the fact that to each ε there corresponds $N = N(\varepsilon, x)$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $n > N$. The notion of *uniform convergence* is simply the removal of the x dependence from N.

DEFINITION 0.1.1. We say that the sequence $\{f_n\}$ *converges uniformly to* f on the domain D if for each and every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$|f(x) - f_n(x)| < \varepsilon$$

holds for all $n > N$ and all x in D.

Remark. Note that the phrase "all x in D" would be superfluous if we replaced $|f(x) - f_n(x)|$ by $\sup_{x \in D} |f(x) - f_n(x)|$. Note also that only trivial modifications are necessary to extend the definition to functions $f = f(x, t)$.

DEFINITION 0.1.2. We say that a sequence $\{f_n\}$ is *uniformly Cauchy* on the domain D if for each and every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$|f_m(x) - f_n(x)| < \varepsilon$$

for all $m, n > N$ and all x in D.

Theorem 0.1.1. *There exists a continuous function f which is the unique uniform limit of the uniformly Cauchy sequence $\{f_n\}$ on the domain D.*

Proof. We give the proof for $D \subset \mathbb{R}$. The unicity of the limit function f follows from the unicity of the $\lim_{n \to \infty} f_n(x)$ of the Cauchy sequence of real numbers $f_n(x)$. The uniformity of the convergence follows from allowing m to tend to infinity in the expression $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n > N(\varepsilon)$ and all x in D. It remains only to show that the limit function is continuous. Select $n_1 > N(\varepsilon/3)$ so that

$$|f(x) - f_{n_1}(x)| < \frac{\varepsilon}{3} \qquad \text{for all } x \in D.$$

As f_{n_1} is continuous, it follows that to $\varepsilon/3$ there corresponds a $\delta = \delta(f_{n_1}, x_0, \varepsilon/3)$ such that

$$|f_{n_1}(x) - f_{n_1}(x_0)| < \frac{\varepsilon}{3}$$

for all $x \in D \cap I(x_0, \delta(f_{n_1}, x_0, \varepsilon/3))$, where $I = \{x \mid x_0 - \delta < x < x_0 + \delta\}$. Thus,

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f_{n_1}(x)| + |f_{n_1}(x) - f_{n_1}(x_0)| \\ &\quad + |f_{n_1}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

holds for all $x \in D \cap I(x_0, \delta(f_{n_1}, x_0, \varepsilon/3))$. As x_0 and ε are arbitrary, f is continuous. □

By the limit (or sum) of a series of functions f_k defined on a common domain D, we mean

$$f(x) = \lim_{n \to \infty} s_n(x),$$

where

$$s_n(x) = \sum_{k=1}^{n} f_k(x).$$

Symbolically, we write

$$f(x) = \sum_{k=1}^{\infty} f_k(x),$$

and say that the series is convergent.

By absolute convergence, we mean $\lim_{n \to \infty} s_n(x)$ exists where

$$s_n(x) = \sum_{k=1}^{n} |f_k(x)|.$$

Uniform and *absolute uniform* convergence are notions that follow directly from the definitions for sequences.

We turn now to the notions of *uniformly bounded* and *equicontinuity*. Sometimes uniformly bounded is replaced by the phrase *equibounded*.

DEFINITION 0.1.3. A sequence of functions f_n is *uniformly bounded* if there exists a positive constant C such that $|f_n(x)| < C$ for all $x \in D$ and all n.

DEFINITION 0.1.4. A sequence of functions f_n is equicontinuous if at each and every $x_0 \in D$ for each and every $\varepsilon > 0$ there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that for all n

$$|f_n(x) - f_n(x_0)| < \varepsilon$$

for all $x \in D \cap I(x_0, \delta(x_0, \varepsilon))$.

We can state now the Ascoli–Arzela Convergence Theorem.

Theorem 0.1.2 (Ascoli–Arzela). *From every uniformly bounded equicontinuous sequence of functions defined on a compact domain, a uniformly Cauchy subsequence of functions can be selected.*

Proof. We give the proof for the domain $[a, b]$. The crux of the proof is the fact that $\mathbb{R} = \overline{\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rational numbers. Likewise, $\mathbb{R}^2 = \overline{\mathbb{Q}}^2$. Both $\mathbb{Q}$ and $\mathbb{Q}^2$ are countable. We return now to $[a, b]$ and let $\{r_n\}$ denote a list of the rationals in $[a, b]$. We note that, given any x in $[a, b]$, there exists a subsequence $\{r_{nk}\}$ such that $\lim_{k \to \infty} r_{nk} = x$.

Now, we consider $\{f_n(r_1)\}$. This sequence of real numbers is bounded via the uniform boundedness assumption. Hence, the Bolzano–Weierstrass theorem implies that we can select a Cauchy subsequence $\{f_{nk}(r_1)\}$ from $\{f_n(r_1)\}$. Rename this subsequence $\{f_{1k}\}$ and consider $\{f_{1k}(r_2)\}$. By the same reasoning we can select a Cauchy subsequence $\{f_{2k}(r_2)\} \subset \{f_{1k}(r_2)\}$. Continuing in this fashion, we obtain a sequence of sequences $\{f_{mk}\}$, $m = 1, 2, 3, \ldots$, such that $\{f_{mk}\} \subset \{f_{m-1k}\}$ and $\{f_{mk}(r_m)\}$ is a Cauchy sequence of real numbers. We define now the subsequence of our theorem by the choice $\{f_{kk}\}$ of the diagonal of the sequences written as a matrix array. Since for fixed m, $m = 1, 2, \ldots$,

$$\{f_{kk}(r_m)\}_{k=m}^{\infty} \quad \text{is a subsequence of} \quad \{f_{mk}(r_m)\}_{k=1}^{\infty},$$

we know that $\{f_{kk}(r_m)\}_{k=1}^{\infty}$ is Cauchy and converges to the same limit; that is,

$$\lim_{k\to\infty} f_{kk}(r_m) = \lim_{k\to\infty} f_{mk}(r_m).$$

Hence, $\{f_{kk}(r_m)\}$ is a Cauchy sequence for each r_m.

We shall use the equicontinuity now to show that $\{f_{kk}(x)\}$ is uniformly Cauchy for all $x \in [a, b]$. Let $\delta = \delta(x, \varepsilon/3)$ denote the δ for $\varepsilon/3$ and x in the definition of equicontinuity. Consider the open intervals

$$I(r_m, \delta(r_m, \varepsilon/3)), \qquad \text{where} \qquad m = 1, 2, 3, \ldots$$

Clearly, $[a, b] \subset \bigcup_{m=1}^{\infty} I(r_m, \delta(r_m, \varepsilon/3))$. Since $[a, b]$ is compact, the Theorem of Heine–Borel implies that there exist p rationals r_i, $i = 1, \ldots, p$ such that $[a, b] \subset \bigcup_{i=1}^{p} I(r_i, \delta(r_i, \varepsilon/3))$. Hence, for any $x \in [a, b]$, there exists an i_0 such that $x \in I(r_{i_0}, \delta(r_{i_0}, \varepsilon/3))$. Thus,

$$\begin{aligned} |f_{mm}(x) - f_{nn}(x)| &\le |f_{mm}(x) - f_{mm}(r_{i_0})| + |f_{mm}(r_{i_0}) - f_{nn}(r_{i_0})| \\ &\quad + |f_{nn}(r_{i_0}) - f_{nn}(x)| \\ &\le \tfrac{2}{3}\varepsilon + |f_{mm}(r_{i_0}) - f_{nn}(r_{i_0})| \end{aligned}$$

by the equicontinuity of the original sequence of functions. Since each sequence $\{f_{kk}(r_i)\}$, $i = 1, \ldots, p$, is a Cauchy sequence, for $\varepsilon > 0$ there exists $N_i = N_i(\varepsilon/3)$ such that

$$|f_{mm}(r_i) - f_{nn}(r_i)| < \frac{\varepsilon}{3}$$

for all m and $n > N_i$. Selecting $N = N(\varepsilon) = \max_{1 \le i \le p} N_i(\varepsilon/3)$, it follows that

$$|f_{mm}(r_i) - f_{nn}(r_i)| < \frac{\varepsilon}{3}$$

for all $i = 1, \ldots, p$, when $m, n > N$. Thus,

$$|f_{mm}(x) - f_{nn}(x)| < \varepsilon$$

holds for all $x \in [a, b]$ when $m, n > N = N(\varepsilon)$. □

We conclude this section with the statement of the Weierstrass Approximation Theorem.

Theorem 0.1.3 (Weierstrass Approximation Theorem). *Let u be a continuous function over the compact domain D. For each $\varepsilon > 0$, there exists a polynomial p_ε such that*

$$|u(s) - p_\varepsilon(s)| < \varepsilon$$

for all $s \in D$.

Proof. We discuss the proof for u defined on $[0, 2\pi]$. First we extend u outside of $[0, 2\pi]$ by setting $u(x) \equiv u(0)$, $x < 0$, and $u(x) \equiv u(2\pi)$, $x > 2\pi$. Next, we can let

$$v(x, h) = h^{-1} \int_x^{x+h} u(\xi)\, d\xi$$

and note that u is the uniform limit of v as h tends to zero since u is uniformly continuous. Likewise, v is the uniform limit of

$$w(x, k) = k^{-1} \int_x^{x+k} v(\xi, h)\, d\xi$$

for each fixed h. Thus, we can assume that u is twice continuously differentiable in $[0, 2\pi]$. For u this smooth, its Fourier series converges absolutely and uniformly. Thus,

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Hence, u can be approximated uniformly by

$$\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

for some large N. Since

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad \text{and} \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2K)!},$$

we truncate the series for $\sin nx$ and $\cos nx$ for $n = 1, \ldots, N$, and obtain a polynomial that approximates u uniformly over $[0, 2\pi]$. □

0.2. THE LEBESGUE DOMINATED-CONVERGENCE THEOREM, LEIBNIZ'S RULE, AND FUBINI'S THEOREM

The Lebesgue Dominated-Convergence Theorem allows us to ignore uniform-convergence hypotheses in exchanging the limit with the integral sign. Here, the integral sign is that of the Lebesgue integral, which is an extension of the Riemann integral. Since the Lebesgue integral coincides with the Riemann integral wherever the latter exists, the reader can view this theorem as a powerful tool for exchanging the limit with the integral sign for Riemann integrals from the calculus.

We state the results for a sequence of functions defined on $[a, b]$.

Theorem 0.2.1 (Lebesgue Dominated-Convergence Theorem). *Let $\{f_n\}$ denote a sequence of integrable functions on $[a, b]$ such that $f(x) = \lim_{n\to\infty} f_n(x)$. Suppose that there exists a positive-valued integrable function g such that $|f_n(x)| < g(x)$ for all $x \in [a, b]$ and all $n = 1, 2, \ldots$ Then the limit function $f(x)$ is integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Remark. For the application of this result the reader need only check that the f_n are Riemann-integrable and that $g(x)$ is Riemann-integrable; this implies the Lebesgue integrability of the f_n and g, from whence it follows that f is Lebesgue-integrable and the limits of the Lebesgue integrals are as stated. Quite often, it can be seen directly that the limit function f is Riemann-integrable. Hence, the great utility of the theorem in its application to Riemann integrals of continuous functions is the ability to ignore hypotheses of uniform convergence of the sequence f_n.

We make two applications of the Lebesgue Dominated Convergence Theorem.

Theorem 0.2.2. *Suppose that $F = F(x, t)$ is defined on $[a, b]\times[\alpha, \beta]$ such that for each $t \in [\alpha, \beta]$, $F(x, t)$ is an integrable function of x, and that for each $x \in [a, b]$, $F(x, t)$ is continuous in t over $[\alpha, \beta]$. Suppose that, for all $t \in [\alpha, \beta]$, $|F(x, t)| \le h(x)$ for some nonnegative integrable function h. Then, the function*

$$H(t) = \int_a^b F(x, t)\,dx$$

is continuous over $[\alpha, \beta]$.

Proof. For $t_0 \in [\alpha, \beta]$, we need only show that $\lim_{t\to t_0} H(t) = H(t_0)$, which is simply

$$\lim_{t\to t_0} \int_a^b F(x, t)\,dx = \int_a^b \lim_{t\to t_0} F(x, t)\,dx.$$

However, this follows immediately from Theorem 0.2.1. □

The final application is Leibniz's rule.

Theorem 0.2.3 (Leibniz's Rule). *Suppose that $F = F(x, t)$ is defined on $[a, b]\times[\alpha, \beta]$ such that, for each $t \in [\alpha, \beta]$, $F(x, t)$ is an integrable function of x and that for each x, $(\partial F/\partial t)(x, t)$ exists and is continuous. Suppose that* *for all* $t \in [\alpha, \beta]$,

$$\left|\frac{\partial F}{\partial t}(x,t)\right| \le g(x)$$

for some nonnegative integrable function g. *Then,* $G(t) = \int_a^b F(x,t)\,dx$ *is differentiable and*

$$G'(t) = \int_a^b \frac{\partial F}{\partial t}(x,t)\,dx$$

for $t \in [\alpha, \beta]$.

Proof. The difference quotients $h^{-1}\{F(x,t+h) - F(x,t)\}$ are integrable since F is and, by the Mean-Value Theorem,

$$|h^{-1}\{F(x,t+h) - F(x,t)\}| = \left|\frac{\partial F}{\partial t}(x,\xi)\right| \le g(x).$$

Hence, by Theorem 0.2.1, $(\partial F/\partial t)(x,t)$ is integrable and

$$\begin{aligned} G'(t) &= \lim_{h \to 0} h^{-1}\{G(t+h) - G(t)\} \\ &= \lim_{h \to 0} \int_a^b h^{-1}\{F(x,t+h) - F(x,t)\}\,dx \\ &= \int_a^b \frac{\partial F}{\partial t}(x,t)\,dx. \end{aligned}$$

□

Remark. We shall leave to the reader the formulation of hypotheses which allow us to compute the derivative of

$$G(t) = \int_a^{\varphi(t)} F(x,t)\,dx$$

and conclude that

$$G'(t) = F(\varphi(t),t)\varphi'(t) + \int_a^{\varphi(t)} \frac{\partial F}{\partial t}(x,t)\,dx.$$

Finally, we wish to recall for the reader Fubini's Theorem on the interchange of the order of integration. We state here a version of Fubini's Theorem that is applicable below.

Theorem 0.2.4. *Let* $f = f(x,y)$ *denote an integrable function on the rectangle* $D = \{(x,y): a \le x \le b, c \le y \le d\}$. *If one of the following integrals exists, then the other two exist and*

$$\int_D\int f\,dx\,dy = \int_c^d \left\{\int_a^b f(x,y)\,dx\right\} dy = \int_a^b \left\{\int_c^d f(x,y)\,dy\right\} dx.$$

Remark. A useful application of Fubini's Theorem is the result

$$\int_0^t\left(\int_0^\tau g(\tau,\eta)\,d\eta\right)dt=\int_0^t\left(\int_\eta^t g(\tau,\eta)\,d\tau\right)d\eta,$$

where in Theorem 0.2.4 we set $a=c=0$, $b=d=t$, $x=\tau$, $y=\eta$, and

$$f(\tau,\eta)=\begin{cases}0, & \eta>\tau,\\ g(\tau,\eta), & \eta\le\tau.\end{cases}$$

0.3. COMPLEX ANALYSIS

We assume that the reader is familiar with the complex numbers $a+ib$, where a and b are real and $i=\sqrt{-1}$ and $|a+ib|=\sqrt{a^2+b^2}$. We summarize here some of the notions and results for analytic functions of the complex variable $z=x+iy$. We let $f=f(z)$ denote a complex-valued function defined in a domain D of the x, y-plane. Thus,

$$f(z)=u(x,y)+iv(x,y),$$

where u and v are real-valued functions of x and y. A function $f=f(z)$ is *analytic* in D if $f'(z)$ exists at each point of D. Here, we take D to be an open path-connected set in the x, y-plane. Computing the derivative via the difference quotient,

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z},$$

leads to the Cauchy–Riemann partial-differential equations. First, with $\Delta z=\Delta x$, we see that

$$f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y).$$

Second, with $z=i\Delta y$, we obtain

$$f'(z)=\frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y).$$

Equating these two results, we obtain the Cauchy–Riemann system

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},\tag{0.3.1}$$

which constitutes a necessary condition on the real and imaginary parts of f for the existence of $f'(z)$. On the other hand if u and v possess continuous first partial derivatives in D and satisfy (0.3.1), then $f'(z)$ exists and is continuous at each point of D.

We shall see that f (and hence) u, and v are infinitely differentiable. Consequently,

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0$$

in D and likewise for $\Delta v = 0$. The solutions of Laplace's equation $\Delta u = 0$ are called *harmonic* functions. The *Dirichlet* problem is the problem of finding a solution of

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases}$$

where f is given on ∂D. The reader is referred to the literature for the discussion of this problem. We note, however, that a bounded solution u satisfies a *weak maximum principle*. Namely,

$$\sup_D u \le \sup_{\partial D} f.$$

We shall make use of this fact below.

We recall Green's theorem, which states that

$$\int_D\!\!\int \left\{ \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right\} dx\,dy = \oint_{\partial D} Q\,dx + P\,dy,$$

where if ∂D is a closed simple (no loops), rectifiable (finite-length), Jordan curve given by $z = z(t) = x(t) + iy(t)$, $0 \le t \le 1$, then the line integral can be computed

$$\oint_{\partial D} Q\,dx + P\,dy = \int_0^1 \{ Q(x(t), y(t))\dot{x}(t) + P(x(t), y(t))\dot{y}(t) \}\,dt.$$

Applying Green's theorem to $\oint_{\partial D} f(z)\,dz$ we see that, for f analytic in D,

$$\begin{aligned} \oint_{\partial D} f(z)\,dz &= \oint_{\partial D} (u + iv)(dx + i\,dy) \\ &= \oint_{\partial D} (u\,dx - v\,dy) + i\oint_{\partial D} (v\,dx + u\,dy) \\ &= \int_D\!\!\int \{ -u_y - v_x \}\,dx\,dy + i\int_D\!\!\int \{ -v_y + u_x \}\,dx\,dy = 0 \end{aligned}$$

since u and v satisfy (0.3.1). We have obtained Cauchy's Theorem.

Theorem 0.3.1 (Cauchy). *For $f = f(z)$ analytic in the closure of a domain D with rectifiable boundary ∂D,*

$$\oint_{\partial D} f(z)\,dz = 0.$$