The Cauchy Problem

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The Cauchy Problem

H. O. Fattorini
University Of California, Los Angeles

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN - CARLO ROTA

Foreword

The Cauchy problem (whose name was coined by Jacques Hadamard in his classical treatise *Lectures on Cauchy's Problem in Linear Partial Differential Equations* published in the Silliman Lecture Series by Yale University Press in 1921) is one of the major problems of the theory of partial differential equations, both in its classical form as it arose in the late nineteenth and early twentieth centuries and in the modern theory, which has seen such a meteoric development since the Second World War. In the classical period, it appeared in two significantly different forms: first as the basic formulation for the most fundamental result in the theory of partial differential equations in the analytic domain—the Cauchy–Kowalewski theorem—as well as the classical boundary value problem, which was relevant to the study of both the wave equation and the more general class of second-order equations of hyperbolic type. In the Cauchy–Kowalewski theorem, the basic local existence theorem for a general (or in the classical case, general second-order) partial differential equation in analytic form with the highest normal derivative near a point on a surface written in terms of derivatives of lower normal-order is given in terms of the Cauchy data—that is, the prescription of the lower normal derivatives on the surface. The Cauchy problem for the wave equation is solvable globally (i.e., for the whole space, or at the very least a non-microscopic region) in terms of Cauchy data on a non-characteristic surface.

The present volume is devoted to an extensive development and exposition of the application of the concept of the *abstract Cauchy problem* to partial differential equations. It might therefore be of value for the relatively uninitiated reader to have stated in a simple and nontechnical form, the reasons that have motivated this concept and that make it worthwhile to study the *abstract Cauchy problem* even in the domain of partial differential equations in which the *concrete Cauchy problem* still plays a major role.

The reasons appear in the discussions of Hadamard's book (and his papers that it summarizes) where for the first time a clear view was developed of the major thrust of the modern theory of boundary value problems for partial differential equations. During the last four decades of the nineteenth century both mathematicians and mathematical physicists (the two fields not being fundamentally distinct at that time) devoted a great deal of effort of the highest degree of originality and technical power to the study of a number of classical boundary value problems for partial differential equations of the greatest significance in mathematical physics, complex function theory, and differential geometry. These include in particular the Dirichlet problem for the Laplace equation, the initial-boundary value problem for the heat equation, and the Cauchy problem for the wave equation. Yet it seems justified to say that the significance of these results for a general theory of partial differential equations and a corresponding theory of boundary value problems was first clarified by Hadamard's very simple criterion for the well-posedness of boundary value problems: that solutions should exist for general classes of reasonable non-analytic data, that they should be uniquely determined by data in such classes, and continuous in appropriate norms on the data and the solutions. Hadamard pointed out that many non-classical problems could be easily shown to be *ill-posed*. It is to this criterion and the corresponding analysis of partial differential operators in terms of their characteristic forms that we owe the basic principles of classification of partial differential equations, a principle clearly absent in the fundamental result in the analytic case—the theorem of Cauchy–Kowalewski.

We must look beyond these remarks (which of course appear in every contemporary textbook on partial differential equations) to a conceptual consequence of considerable historical interest. Hadamard's remark was essentially the application to the concrete analytic area of partial differential equations of a fundamental principle from another developing area of mathematical study: functional analysis. His criterion could be most clearly stated in the general language: Examine the mapping or operator T, which assigns to each datum element f the corresponding solution u of the boundary value problem. Then the problem is well-posed if $u = T(f)$ is well-defined for each f, and continuous from the space of data (appropriately normed) to the space of solutions.

The abstract Cauchy problem is both a refinement and a more specific form for the direction of thought implied by Hadamard's criterion. Whatever the formal dates may be for its detailed formulation, it seems to have arisen between the two World Wars as a response to developments in two originally distinct mathematical domains. Most explicitly, it appears in the study of the initial-value problem for the Schrödinger equation, which is central in the Schrödinger formulation of the quantum theory, an ordinary differential equation of first order for an unknown function u from the real line to an infinite-dimensional Hilbert space. The initial-value problem for this equation,

$$\frac{du}{dt} = \frac{i}{\hbar} H(u),$$

gives a paradigm for the study of such initial-value problems in infinite-dimensional spaces. The corresponding development in the 1930s and beyond of the theory of Markoff processes and more general stationary random processes with such basic principles as the semi-group law embodied in the Chapman–Kolmogoroff equations gave rise to strong motives for the post-war study of the theory of one-parameter semi-groups—that is, maps from the non-negative real numbers t in R^+ to operators $U(t)$ in such infinite-dimensional spaces as $C(\Omega)$ or $L^1(d\mu)$ satisfying the semi-group law $U(t)U(s) = U(t+s)$. The central theme of this study was the determination of the infinitesimal generator of the semi-group—that is, an operator A such that $U(t)f = u(t)$ could be characterised as the solution (at least for suitably nice f) of the differential equation $du/dt(t) = Au(t)$, with $u(0) = f$. Conversely, the theory of analytic semi-groups [and its extension to the corresponding theory of evolution equations that generalize the semi-group law to the corresponding composition conditions that are valid for ordinary differential equations

$$\frac{du}{dt} = A(t)u(t)]$$

developed a sophisticated apparatus for the generation of semi-groups by linear operators A (usually closed and densely defined but not continuous), which is a significant branch of the modern theory of linear functional analysis. More recently still, there has been a very flourishing development of a corresponding theory for nonlinear operators A and nonlinear mappings $U(t)$, the latter being non-expansive or Lipschitzian mappings, which has extended a significant part of the linear theory to the nonlinear domain.

The present volume by Professor Fattorini appears at an interesting juncture in this process of development. The paradigm of the theory of one-parameter semi-groups of linear mappings and their relation to their infinitesimal generators has reached and passed its apogee. Even the nonlinear theory has probably achieved its principal goals as far as very general results are concerned. Yet the application of this now-classical machinery to the basic core of problems concerning the classical equations of mathematical physics has not been thoroughly exposed and digested in the expository literature available to nonspecialists and the mathematically interested in various domains of science and engineering. One needs a form of exposition to use in the analysis of applied problems rather than on the frontier of technical research in this area of functional analysis.

Professor Fattorini's book makes an important contribution to this process. Its central concern is with the equations of mathematical physics. It spends relatively little space on the other broad theme of application—the theory of stochastic processes. This is perfectly natural, however, since the nature of the terrain of these two major domains of application have such significant conceptual and technical differences. One may hope with such treatments as this book presents that the relatively completed paradigm of the abstract Cauchy problem will become an easily usable and well-understood instrument in the domains for which it was designed to apply.

FELIX E. BROWDER

Preface

Consider an initial value or initial-boundary value problem

$$u_t = Au, \quad u = u_0 \quad \text{for} \quad t = 0, \tag{1}$$

where A is, say, a partial differential operator in the "space variables" $x_1, \ldots, x_m$. It was discovered independently by E. Hille and K. Yosida about forty years ago that (1) can be studied to advantage under the form of an "ordinary differential" initial value problem

$$u'(t) = Au(t), \quad u(0) = u_0, \tag{2}$$

where now A is thought of as an operator in a suitable function space E, u is a function of t taking values in E, and the boundary conditions (if any) are included in the definition of the space or of the domain of A. The formal similarity of (2) with a system of ordinary differential equations provides us with heuristic insight on the original problem; for instance, we may expect to be able to write the solution of (2) (thus of (1)) in the form $u(t) = \exp(tA)u_0$ (this is in fact true if the exponential is correctly interpreted). Moreover, results on (2) may apply to many different types of partial differential equations or even to more general equations, an expectation that is borne out in practice.

A large body of theory along these lines was developed during and after the forties and now permeates most advanced treatments of hyperbolic and parabolic initial-boundary value problems; it applies equally well to equations not purely differential, such as the neutron transport equation. Highlights of this development have been the introduction of dissipative operators by R. S. Phillips and the extension of many of the basic results to operators A depending on t by T. Kato and H. Tanabe. Finally, the theory of the initial value problem (2) has been extended in the last twenty years to include nonlinear equations; this has proved to be a deep and fruitful field where important research is taking place even today. Along the way, the theory of abstract differential equations and its equivalent formulation—semigroup theory—have found significant applications in many areas; among the most recent have been singular perturbations and control theory. We may also mention that semigroup theory has been an essential language in such computational developments as finite difference methods for partial differential equations.

Nowadays, many volumes devoted wholly or partly to the treatment of semigroup theory exist, foremost among them the encyclopedic treatise of Hille and Phillips, still the standard reference in many areas. In contrast, accounts of the applications to particular partial differential and other equations are scarcer, usually being part of treatises on partial differential equations. Other basic applications are only found within the research literature and are for this reason not readily accessible to nonspecialists.

I have attempted to bridge this gap, at least partly, in the present volume by collecting some basic results on the equation (2) and on its time-dependent version that can be readily applied to a variety of equations and are (or may be suspected to be) in a reasonably definitive form. Most of the material presented is on applications of these results. Anything resembling completeness in so vast a field is of course out of the question, but I hope the wide range of examples presented will provide the reader with fairly general and useful ideas on how to fit an equation like (1) into the mold of abstract differential equations, and on what the general results mean when applied to particular equations. A specialist may find here and there (perhaps in the large bibliography) some new facts; however, the intended audience for this book is scientists, engineers, and applied mathematicians looking for efficient ways to handle particular problems.

The prospective reader is expected to have some familiarity with ordinary differential equations and a good knowledge of real variable theory, in particular the Lebesgue integral and Lebesgue spaces; an acquaintance with complex variable at the undergraduate level is sufficient. Also, a knowledge of elementary functional analysis is necessary; most of what is needed is included, partly without proofs, in the introductory Chapter 0, although the only indispensable information there is that on resolvents of unbounded operators. No familiarity with the theory of partial differential equations is assumed (except in some parts of Chapters 1 and 4); however, some information on the classical equations (Laplace, wave, heat) may help to put results in perspective. Within these requirements this book is essentially self-contained with the exception of numerous results on Sobolev spaces and two on singular integrals, which are stated without proof in Chapter 4. Some distribution theory is used through the text; the few facts on vector-valued distributions needed in Chapter 8 are presented there with complete proofs.

The contents of this book can be described as follows. Chapter 1 consists almost exclusively of examples drawn from problems of mathematical physics; the resulting equations are treated by ad hoc methods (Fourier series and transforms) and the results provide motivation for the definition of well-posed Cauchy problem. The resulting theory is examined in Chapter 2. Chapter 3 discusses the particular case corresponding to dissipative operators and some related facts (such as semigroups in Banach lattices) with applications to second order ordinary differential operators and symmetric hyperbolic equations. Chapter 4 is on abstract parabolic equations (chiefly the analytic case) and on applications to second order parabolic equations. Chapter 5 deals with perturbation theory; the applications include the neutron transport equation and the Schrödinger and Dirac equations with potentials. Other topics include continuous and discrete approximations to abstract differential equations, among them finite difference methods, which are illustrated with a parabolic initial-boundary value problem. Further considerations on the idea of well posed problem are found in Chapter 6, where formulations different from that of Cauchy problem are introduced in several examples. The theory of the equation (2) with A depending on t is the subject of Chapter 7. Finally, we present in Chapter 8 a brief account of the theory of the Cauchy problem in the sense of distributions. Here the restriction to purely differential equations of the type of (2) is unreasonable and we consider instead hereditary equations; relations with system theory are pointed out. In the case of the equation (2) this formulation is seen to be equivalent to the so-called mildly well-posed Cauchy problem, brought into existence as a tool for the treatment of hyperbolic equations with multiple characteristics.

Some shortcuts through the text will be evident to the reader. For instance, Chapter 1 may be passed over except for the definition of well-posed Cauchy problem and for some results on self-adjointness of the unperturbed Schrödinger and Dirac operators to be used in Chapter 5. Chapters 2, 3, and the first two sections of Chapter 4 are basic for the understanding of most of the subsequent material, as are Sections 1 and 3 of Chapter 5; the rest of these chapters and the remaining ones are fairly independent of each other.

Numerous paragraphs labelled "Example" can be found throughout the text. Some of these are worked out in detail; others are not and should be understood as exercises. The most difficult ones are starred and references are given.

Sections and paragraphs in small type can be omitted without detriment to the comprehension of subsequent material.

Each chapter ends with "Miscellaneous" comments; these give historical information, discuss parts of the theory not treated in detail, and provide bibliographical indications.

The selection of topics reflects of course the tastes and limitations of the author. Without a doubt, the most important subject we have not covered is that of nonlinear equations (although some references to quasilinear equations are found in Chapter 2). Even within the field of linear equations much has been omitted or is barely mentioned in passing. Some of these topics are: applications of semigroup theory to probability, random evolutions and differential operators of order greater than two, as well as the theory of the equation (2) not directly associated with the well posed Cauchy problem.

I am happy to acknowledge my indebtedness to numerous individuals and institutions in relation to the writing of this book. A set of lecture notes that became part of Chapters 2, 3, 4, and 5 was written during 1972 and 1973 at the Universidad de Buenos Aires, in preparation for a projected course on applications of functional analysis. My thanks go to my colleagues there for making possible a fruitful stay, which would not have been viable without the support of the Consejo Nacional de Investigaciones Científicas y Técnicas of Argentina. Likewise, I am grateful to my colleagues at the Università di Firenze and to the Consiglio Nazionale delle Richerche of Italy for their arrangements for my visit during the spring of 1975, at which time the actual writing began.

Finally, I wish to point out that the completion of this book would have been impossible without the effective and understanding support of the National Science Foundation given during the entire period of its preparation.

To my wife Natalia go my sincere thanks for her patience and understanding, as well as for her constant encouragement.

H. O. FATTORINI

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