Chapter 0

Elements of Functional Analysis

1. BANACH AND HILBERT SPACES. BOUNDED OPERATORS. INTEGRATION OF VECTOR-VALUED FUNCTIONS

We shall denote by $E, F, X, Y, \ldots$ etc. real or complex Banach spaces; a Hilbert space will be named H. Unless otherwise stated all spaces will be assumed *complex*. We denote by $\|\cdot\|_E$ the norm in a Banach space E, by $(\cdot,\cdot)_H$ the scalar product in a Hilbert space H, and by $(E; F)$ the space of all linear bounded operators from E into F endowed with the norm $\|A\| = \sup\{\|Au\|_F; \|u\|_E \le 1\}$ (the "uniform operator norm"). Clearly we have $\|Au\|_F \le \|A\|_{(E;F)}\|u\|_E$; also, if $A \in (F; X)$ and $B \in (E; F)$, then $AB \in (E; X)$ and $\|AB\|_{(E;X)} \le \|A\|_{(F;X)}\|B\|_{(E;F)}$. When confusion is not likely (that is, nearly always), we shall drop the subindices from norms and scalar products; also $(E; E)$ sometimes will be abbreviated to (E).

The following two results are basic.

1.1 Banach-Steinhaus Theorem. *Let $\{A_n\}$ be a sequence in $(E; F)$ such that* (a) $\|A_n\| \le C$ *for all n.* (b) $\lim A_n u$ *exists for u in a dense subset of E. Then $Au = \lim A_n u$ exists for all $u \in E$, $A \in (E; F)$, and $\|A\| \le \liminf \|A_n\|$.*

1.2 Uniform Boundedness Theorem. *Let $\mathfrak{B}$ be a subset of $(E; F)$ such that $\{Bu; B \in \mathfrak{B}\}$ is bounded in F for each $u \in E$. Then $\|B\| \le C$ $(B \in \mathfrak{B})$; that is, $\mathfrak{B}$ is bounded in $(E; F)$.*

For proofs see Hille-Phillips [1957: 1, pp. 41 and 26].

We recall that invertible operators are an open subset of $(E; F)$. In fact, if $A \in (E; F)$ possesses an inverse $A^{-1} \in (F; E)$, then so does every operator $B \in (E; F)$ with $\|B - A\| < \|A^{-1}\|^{-1}$ and

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} \left((A - B)A^{-1}\right)^n \in (F; E), \tag{1.1}$$

the series convergent in the norm of $(F; E)$ in the range of B indicated above.

Given $A \in (E)$, the *resolvent set* of A (written ρ or $\rho(A)$) is the set of all complex λ such that $\lambda I - A$ (I the identity operator in E) has an inverse $R(\lambda) = (\lambda I - A)^{-1} \in (E)$, which is called the *resolvent* or *resolvent operator* of A. The complement $\sigma = \sigma(A)$ is the *spectrum* of A. As a consequence of our previous observations on inverses, we deduce that $\rho(A)$ is always open; precisely, if $\lambda \in \rho(A)$, then every μ with $|\mu - \lambda| \le \|R(\lambda)\|^{-1}$ belongs to $\rho(A)$ and

$$R(\mu) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda)^{n+1}, \tag{1.2}$$

the series convergent in the indicated range; moreover, $\rho(A)$ contains the set $\{\lambda; |\lambda| > \|A\|\}$ and

$$R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n. \tag{1.3}$$

In particular, $\rho(A)$ is never empty and $\sigma(A)$ is compact. It can be shown (Hille-Phillips [1957: 1, p. 125]) that $\sigma(A)$ is nonempty as well.

If E is a real Banach space, all the previous statements have an immediate counterpart, except that $\sigma(A)$ may be empty (this can be exemplified by a real matrix without real eigenvalues).

Back in the general case, let f be a function defined in a compact interval $a \le t \le b$ and taking values in E. If f is continuous (or only piecewise continuous), the Riemann integral of f can be defined and shown to exist in the same way as in the scalar case as a limit of Riemann sums,

$$\int_a^b f(t)\,dt = \lim \sum_{k=1}^{n} (t_k - t_{k-1}) f(\xi_k),$$

the limit understood in the norm of E. Integrals over multidimensional sets are similarly defined, and improper Riemann integrals (that is, integrals over noncompact sets) are obtained in the usual way as limits of integrals on a convenient sequence of compact subsets. All the properties of ordinary Riemann integrals that make sense in the vector-valued setting can be proved by rather trivial modifications of the classical proofs. An important existence criterion for improper integrals (which, to fix ideas, we state only for the real line) is the following: if J is a (finite or infinite) interval in $\mathbb{R} = (-\infty, \infty)$ and f is an E-valued piecewise continuous function defined in J and such that $\|f(\cdot)\|$ is integrable in J, then f is integrable over J and

$$\left\|\int_J f(t)\,dt\right\| \le \int_J \|f(t)\|\,dt. \tag{1.4}$$

This result holds as well, of course, in multidimensional regions. We note in passing that the fundamental theorem of calculus holds: if f is an E-valued continuous function defined in $a \le t \le b$ and

$$g(t) = \int_a^t f(s)\,ds, \tag{1.5}$$

then

$$g'(t) = f(t) \quad (a \le t \le b), \tag{1.6}$$

the derivative understood as the limit in E of the corresponding quotient of increments. Conversely, if g is continuously differentiable in $a \le t \le b$ with derivative f, then g is given by (1.5) modulo a constant (that is, modulo an element of E that does not depend on t).

Let $A(\cdot)$ be a function defined in an interval J of the real line with values in $(E; F)$. If A is continuous, the previously outlined integration theory can be automatically applied to it. However, we shall have to contend most of the time with functions that are merely *strongly continuous*, that is, such that $t \to A(t)u$ is continuous in F for each $u \in E$. The integral can then be defined "elementwise"; that is, although $\int A(t)\,dt$ may not make sense, $\int A(t)u\,dt$ does for every $u \in E$. We state the following useful result, restricted for definiteness to the real line.

1.3 Lemma. *Let $A(\cdot)$ be strongly continuous in an interval J. Assume that $\|A(\cdot)\|$ is integrable in J. Then the operator defined by*

$$Au = \int_J A(t)u\,dt \tag{1.7}$$

belongs to $(E; F)$; precisely,

$$\|A\| \le \int_J \|A(t)\|\,dt. \tag{1.8}$$

The proof is an immediate consequence of (1.4) and the preceding comments.

We note in passing that the integrand in (1.8) is not necessarily continuous under the present assumptions. However, $\|A(t)\| = \sup\{\|A(t)u\|; \|u\| \le 1\}$, where each $t \to \|A(t)u\|$ is continuous, so that $\|A(\cdot)\|$ is lower semicontinuous. On the other hand, if J' is a compact subinterval of J, $\|A(\cdot)u\|$ is continuous, hence bounded in J' for each $u \in E$. It follows from the uniform boundedness theorem (1.2) that $\|A(\cdot)\|$ is bounded in J'. This gives sense to the integrability assumption in Lemma 1.3.

A rather complete exposition of the theory of Riemann (or, more generally, Riemann-Stieltjes) integration of vector-valued functions can be found in Hille-Phillips [1957: 1, Sec. 3.2] and in Hille [1972: 1, Sec. 7.4]. We shall also use a few times the theory of Lebesgue integrals (as generalized by Bochner) of vector-valued functions, which is in Hille-Phillips [1957: 1, Sec. 3.7] and Hille [1972: 1, Sec. 7.5]. We note that both integration theories can be made to function equally well in a real or in a complex space.

2. LINEAR FUNCTIONALS: THE DUAL SPACE. VECTOR-VALUED ANALYTIC FUNCTIONS

Using the notation in the previous section we define E^*, the *dual* of E, as $E^* = (E;\mathbb{C})$ ($\mathbb{C}$ the complex numbers). The elements of E^* are the bounded linear functionals in E. Application of a functional $u^* \in E^*$ to an element u of E will be denoted $\langle u^*, u\rangle$, $u^*(u)$, or $\langle u, u^*\rangle$. It is clear from the definition of the norm in E^* that the "Cauchy-Schwarz inequality"

$$|\langle u^*, u\rangle| \le \|u^*\|\,\|u\| \tag{2.1}$$

holds for all $u \in E$, $u^* \in E^*$.

The existence of abundant linear functionals in E is assured by the

2.1 Hahn-Banach Theorem. *Let N be a subspace of E, $f: N \to \mathbb{C}$ a linear functional such that $|f(u)| \le C\|u\|$ $(u \in N)$. Then there exists a $u^* \in E^*$ such that* (a) $u^*(u) = f(u)$ $(u \in N)$. (b) $\|u^*\| \le C$.

Theorem 2.1 is actually a very particular case of the general Hahn–Banach theorem whose proof can be found in Banach [1932: 1, p. 27] or Hille-Phillips [1957: 1, p. 29]. We note that the Hahn–Banach theorem holds also in real spaces (actually, this was the case originally proved by Banach).

2.2 Corollary. *Let N be a closed subspace of E, $u \notin E$. Then there exists a functional $u^* \in E^*$ such that $u^*(N) = \{0\}$, $u^*(u) = 1$, $\|u^*\| = 1/d$, where $d = \mathrm{dist}(u, N) > 0$.*

The proof is immediately obtained by defining a functional f in the subspace $\{w = v + \lambda u;\ v \in N, \lambda \in \mathbb{C}\}$ generated by N and u by $f(v + \lambda u) = \lambda$ and applying Theorem 2.1 (see Hille-Phillips [1957: 1, p. 30]). Taking N itself equal to $\{0\}$, we obtain

2.3 Corollary. *Let $u \in E$. Then there exists a $u^* \in E^*$ with*

$$\langle u^*, u\rangle = \|u\|^2 = \|u^*\|^2. \tag{2.2}$$

By virtue of the obvious bilinearity of the function $\langle u^*, u\rangle \to \langle u^*, u\rangle$, an element $u \in E$ gives rise to a $u^{**} \in E^{**} = (E^*)^*$ through the formula

$$\langle u^{**}, u^*\rangle = \langle u, u^*\rangle. \tag{2.3}$$

It is clear that the map $u \to u^{**}$ from E into E^{**} thus obtained is linear; moreover, it follows from (2.1) that $\|u^{**}\|_{E^{**}} \le \|u\|_E$, and from (2.2) that we actually have $\|u^{**}\|_{E^{**}} = \|u\|$. We thus see that E can be identified (linearly and metrically) with a subspace of E^{**}; in a less precise fashion we may say that "E is a closed subspace of E^{**}." If $E = E^{**}$—that is, if every continuous linear functional in E^* can be expressed by the formula (2.3) for some $u \in E$—we say that the space E is *reflexive*.

Let Ω a domain (open connected set) in the complex plane, and $f(\zeta)$ a function defined in Ω with values in E. We say that f is *analytic*, or *holomorphic*, in Ω if the derivative $f'(\zeta)$ exists (as the limit in the norm of E of the corresponding quotient of increments) for all $\zeta \in \Omega$. We can also define analyticity by means of linear functionals as follows: f is analytic in Ω if and only if the scalar function $\langle u^*, f(\cdot)\rangle$ is analytic in Ω for all $u^* \in E^*$. Clearly the first definition implies the second. Remarkably enough, the two are actually equivalent. If $A(\cdot)$ is an $(E; F)$-valued function defined in Ω, the first definition of analyticity (existence of the limit of the quotient of increments in the norm of $(E; F)$) is likewise equivalent to the statement that $\langle u^*, A(\cdot)u\rangle$ is an ordinary analytic function for any $u \in E$ and any $u^* \in F^*$ (for a proof see Hille-Phillips [1957: 1, p. 93]). Vector-valued analytic functions share most properties of ordinary analytic functions. Roughly speaking, the proofs are carried out by imitating the proof for the scalar case (replacing when appropriate modulus by norm) or by applying functionals and then using the scalar theory. We state a few facts that will be of use later.

2.4 Lemma. *Let f be a E-valued analytic function in Ω. Then* (a) *if Γ is a simple closed contour such that Γ and its interior are contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \tag{2.4}$$

if z belongs to the interior of Γ and Γ is oriented counterclockwise. (b) *If $z \in \Omega$ f can be developed in power series about z,*

$$f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - z)^n \tag{2.5}$$

where $a_n = f^{(n)}(z)/n!$ and the series (2.5) *converges absolutely and uniformly in $|\zeta - z| \le d'$ if $d' < d = \mathrm{dist}(z,$ boundary of $\Omega)$ (that is, $\sum\|a_n\||\zeta - z|^n$ converges uniformly in $|\zeta - z| \le d'$).*

2.5 Lemma. *An arbitrary power series* $\Sigma a_n(\zeta - z)^n$ *with coefficients in E converges absolutely and uniformly in* $|\zeta - z| \le d'$ *if* $d' < r$,

$$1/r = \limsup \|a_n\|^{1/n} \tag{2.6}$$

and diverges if $|\zeta - z| > r$. *The function* $f(\zeta) = \Sigma a_n(\zeta - z)^n$ *is analytic in* $|\zeta - z| < r$ *and the successive derivatives are obtained by differentiation of* (2.5) *term by term.*

For the proofs of these and other results the reader may consult Hille-Phillips [1957: 1, Sec. 3.11]. We note the following interesting consequence of Lemma 2.5. Let $A \in (E)$. Define $r = r(A)$, the *spectral radius* of A, by $r = \sup\{|\lambda|;\ \lambda \in \sigma(A)\}$. It follows from (1.2) that $R(\lambda)$ is analytic in $\rho(A)$. Applying then Lemma 2.4 to $R(1/\lambda)$, we deduce that (1.3) must converge for $|\lambda| > r$; thus it results from Lemma 2.5 that

$$r(A) = \limsup \|A^n\|^{1/n}. \tag{2.7}$$

We note finally that the lim sup on the right-hand side of (2.7) is actually a limit; this follows from Pólya-Szego [1954: 1, p. 17, Nr. 98] and from the obvious inequality $\|A^{m+n}\| \le \|A\|^m \|A\|^n$.

A thorough treatment of the theory of vector-valued analytic functions can be found in Hille-Phillips [1957: 1, Ch. 3] and also in Hille [1972: 1, Ch. 8]. We note in particular that the facts concerning Laurent series and development in power series about singular points extend to the present case.

3. UNBOUNDED OPERATORS; THE RESOLVENT. CLOSED OPERATORS

We shall deal most of the time with operators (e.g., differential operators) that are neither bounded nor everywhere defined. We extend then our previous definition of linear operator by allowing A to be defined in a subspace $D(A)$ of a Banach space E and to take values in another Banach space F. Only the case $E = F$ will be used. The subspace $D(A)$ is called the *domain* of A; in most instances $D(A)$ will be dense in E (we say then that A is *densely defined*) but not always. The *sum* of two operators is defined by $(A + B)u = Au + Bu$ with $D(A + B) = D(A) \cap D(B)$ while the *product* or *composition* is $(AB)u = A(Bu)$, where $D(AB)$ consists of all $u \in D(B)$ with $Bu \in D(A)$. (Note that, even if A and B are densely defined, $A + B$ and AB may not be; in fact, both $D(A + B)$ and $D(AB)$ may reduce to $\{0\}$.) If A is bounded ($\|Au\| \le C\|u\|$, $u \in D(A)$) and densely defined, we can extend A by continuity to all of E; we shall then usually assume that all bounded operators are everywhere defined.

An intermediate notion between that of bounded operator and the general definition introduced in this section is that of closed operator. An operator A is *closed* if and only if whenever $\{u_n\}$ is a sequence in $D(A)$ such that $u_n \to u$ and $Au_n \to v$ for some $u, v \in E$, it follows that $u \in D(A)$ and $Au = v$. Clearly a bounded, everywhere defined operator is closed. An equivalent definition of closedness is the following. Consider the Cartesian product $E \times E$ endowed with any norm that imparts to it the product topology (for instance, $\|\{u, v\}\|_{E\times E} = \|u\| + \|v\|$). Then $E \times E$ is a Banach space. The *graph* of the operator A is

$$\Gamma(A) = \{\{u, Au\}; u \in D(A)\}.$$

Clearly $\Gamma(A)$ is always a subspace of $E \times E$ and it is closed if and only if A is closed according to the previous definition.

A fundamental result is:

3.1 Closed Graph Theorem. *Let A be closed and everywhere defined. Then A is bounded.*

This theorem is equivalent to the following result, which is of great importance in its own right:

3.2 Open Map Theorem. *Let $B \in (E; F)$. Assume that $B(E) = F$. Then if U is an open set in E, $B(U)$ is an open set in F.*

3.3 Corollary. *Let $B \in (E; F)$. Assume that B is one-to-one and $B(E) = F$; that is, assume that B has an inverse B^{-1}. Then $B^{-1} \in (F; E)$.*

For proofs of these and more general results, see Banach [1932: 1, p. 38], Hille-Phillips [1957: 1, pp. 46–47], or Dunford-Schwartz [1958: 1, pp. 55–58].

3.4 Lemma. *Let A be a closed operator in E. Assume that $u(\cdot)$ is defined in a (bounded or unbounded) interval J and takes values in $D(A)$; suppose, moreover, that $u(\cdot)$, $Au(\cdot)$ are continuous (or only piecewise continuous) and that $\|u(\cdot)\|$, $\|Au(\cdot)\|$ are integrable in J. Then $\int_J u(t)\,dt \in D(A)$ and*

$$A\int_J u(t)\,dt = \int_J Au(t)\,dt. \tag{3.1}$$

The proof is immediate for compact intervals; in fact, we only have to approximate $\int u\,dt$ by a sequence of Riemann sums $u_n = \sum_k (t_{k,n} - t_{k-1,n})u(\xi_{k,n})$ and observe that Au_n is a sequence of Riemann sums approximating $\int Au\,dt$. If J is not compact, we approximate $\int_J u\,dt$ by $\int_{J_n} u\,dt$, where J_n is a sequence of compact subintervals of J, and apply the previous observation.

Naturally, this result can be greatly generalized; the integral may be multidimensional or understood in the sense of Lebesgue–Bochner. For a very general statement the reader may consult Hille-Phillips [1957: 1, p. 83].

Given an operator A (not necessarily densely defined) in E, we define, as in the bounded case, the *resolvent set* of A (denoted again $\rho(A)$)

as the set of all complex λ such that $\lambda I - A$ has a bounded inverse $R(\lambda) = R(\lambda; A)$. In the present situation this means

$$(\lambda I - A)R(\lambda) = I \tag{3.2}$$

and

$$R(\lambda)(\lambda I - A)u = u \quad (u \in D(A)). \tag{3.3}$$

Note that, in view of the definition of the domain of the composition of two operators, (3.2) includes the requirement that $R(\lambda)E \subseteq D(\lambda I - A) = D(A)$. The spectrum of A, $\sigma(A)$, is again defined as the complement of $\rho(A)$. Unlike as in the case where A is bounded, there are examples where $\rho(A)$ or $\sigma(A)$ are empty, but it is still true that $\rho(A)$ is open and that $R(\cdot)$ is an analytic function in $\rho(A)$. Its Taylor development about $\lambda \in \rho(A)$ is given as before by

$$R(\mu) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda)^{n+1}, \tag{3.4}$$

the series convergent in (E) for $|\mu - \lambda| \leq \|R(\lambda)\|^{-1}$ (in particular, every such μ belongs to the resolvent set $\rho(A)$). For a proof see Dunford-Schwartz [1958; 1, p. 566]. An important ingredient in the proof is the following observation: if $B \in (E; E)$ is one-to-one, then its inverse B^{-1} (which may not be everywhere or even densely defined) is closed. This shows that $\lambda I - A = R(\lambda)^{-1}$ is closed if $\lambda \in \rho(A)$, a fortiori A itself is closed. The converse is not true: a closed operator may very well have an empty resolvent set. It follows immediately from (3.4) that powers and derivatives of the resolvent are related by the following formula:

$$R(\lambda)^{(n)} = (-1)^n n! R(\lambda)^{n+1} \quad (\lambda \in \rho(A), n \geq 1). \tag{3.5}$$

An immediate consequence of formula (3.4) and following comments is:

3.5 Lemma. *Let the sequence $\{\lambda_n\}$ belong to $\rho(A)$; assume that $\lambda_n \to \lambda$ and that $\|R(\lambda_n)\|$ remains bounded (or, more generally, that $\|R(\lambda_n)\| = o(|\lambda - \lambda_n|)$. Then $\lambda \in \rho(A)$.*

In fact, the disk $|\mu - \lambda_n| \leq \|R(\lambda_n)\|^{-1}$ must be contained in $\rho(A)$ for all n; for sufficiently large n the disk contains λ.

In intuitive terms, Lemma 3.5 states that "the resolvent must blow up (rapidly enough) when we approach the spectrum."

The next result, equally elementary, establishes that the resolvent may be constructed by analytic continuation.

3.6 Lemma. *Let Ω be an open connected set in $\mathbb{C}$ such that $\Omega \cap \rho(A) \neq \varnothing$, and let $Q(\lambda)$ be a (E)-valued analytic function in Ω such that $Q(\lambda) = R(\lambda)$ in $\Omega \cap \rho(A)$. Then $\Omega \subseteq \rho(A)$ and $Q(\lambda) = R(\lambda)$ in Ω.*

To prove Lemma 3.6 use Lemma 3.5 as follows. Take $\lambda_0 \in \Omega \cap \rho(A)$, $\lambda \in \Omega$ and join them by a curve Γ in Ω. Let $\Gamma' = \Gamma \cap \rho(A)$. Then Γ' is open in Γ; but $R(\lambda)=Q(\lambda)$ remains bounded in Γ' so that by Lemma 3.5 Γ' must as well be closed in $\rho(A)$. By connectedness $\Gamma'=\Gamma$.

Equalities (3.2) and (3.3) are sometimes called the *first resolvent equation(s)*. *The second resolvent equation*, which results rather easily from the definition, is

$$R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu) \quad (\lambda,\mu\in\rho(A)). \tag{3.6}$$

Another useful equation involving the resolvent can be obtained taking $u\in D(A^m)$, writing (3.3) in the form $R(\lambda)u=\lambda^{-1}u+\lambda^{-1}AR(\lambda)u$, replacing the left-hand side into the right-hand side, and iterating. The result is

$$R(\lambda)u=\frac{1}{\lambda}u+\frac{1}{\lambda^2}Au+\cdots+\frac{1}{\lambda^m}A^{m-1}u$$
$$+\frac{1}{\lambda^m}R(\lambda)A^m u \quad (\lambda\in\rho(A),\lambda\neq 0). \tag{3.7}$$

We have observed in the previous section that if A is bounded, then $R(\lambda)$ exists for $|\lambda|$ large enough. This property is shared by the resolvents of some unbounded operators. However, we have:

3.7 Lemma. *Assume $R(\lambda)$ exists for $|\lambda|>a$ and has a pole at infinity. Then A is bounded.*

In fact, we must have

$$R(\lambda)=\sum_{n=-\infty}^{m}\lambda^n B_n \quad (|\lambda|>a), \tag{3.8}$$

where $B_n\in(E)$ $(-\infty<n\leq m)$. The coefficients are given by the usual formulas

$$B_n=\frac{1}{2\pi i}\int_{|\lambda|=a+1}\lambda^{-(n+1)}R(\lambda)\,d\lambda,$$

and it follows then from Lemma 3.4 that $B_n(E)\subseteq D(A)$. Making use of (3.2) in (3.8) and equating coefficients in the power series thus obtained we arrive at the relations $B_m=0$, $B_n=AB_{n+1}$ $(n\neq-1)$, $B_{-1}=AB_0+I$, which show that $B_0=B_1=\cdots=B_m=0$, $B_{-1}=I$, $B_{-2}=A$, so that $A\in(E)$ as claimed.

3.8 Example. (a) Let $E=L^2(0,1)$ (see Section 7), A the operator

$$Au(x)=u'(x), \tag{3.9}$$

$D(A)$ defined as the set of all $u\in L^2(0,1)$ that are absolutely continuous in $0<t<1$ and such that $u'\in L^2(0,1)$. Then A is densely defined and closed but $\rho(A)=\varnothing$ (the equation $(\lambda I-A)u=v$ has multiple solutions.) (b) Let B_0 be the restriction of A defined by $u(0)=0$. Then B_0 is densely defined and $\rho(B_0) = \mathbb{C}$. $R(\lambda)$ has an essential singularity at infinity. (c) Let B be the restriction of A defined by $u(0) = u(1) = 0$. Then $\rho(B) = \varnothing$ (the equation $(\lambda I - B)u = v$ has no solution except for certain right-hand sides.)

It is sometimes useful to classify complex numbers in the spectrum of an operator. We shall only single out *eigenvalues*—that is, complex numbers λ such that there exists $u \in D(A)$, $u \neq 0$ such that $(\lambda I - A)u = 0$, or

$$Au = \lambda u. \tag{3.10}$$

Solutions $u \neq 0$ of (3.10) are called *eigenvectors* (corresponding to the eigenvalue λ); when λ is an eigenvalue, the set of all solutions of (3.10) is a nontrivial subspace $E(\lambda)$ of E called the *eigenspace corresponding to* λ. If A is closed, its eigenspaces (if any) are closed. Given an eigenvalue λ, a vector $u \in D(A^m)$ is called a *generalized eigenvector* if there exists an integer $m \geq 1$ such that

$$(\lambda I - A)^m u = 0 \tag{3.11}$$

for some $u \in D(A^m)$, $u \neq 0$; *generalized eigenspaces* $E_g(\lambda)$ are defined in the obvious way. A generalized eigenvector has *degree m* if it satisfies (3.11), but $(\lambda I - A)^{m'} u \neq 0$ for $m' < m$; the *degree* of an eigenvalue λ is the maximum of the degree of all its generalized eigenvectors (which may be infinite). Equally infinite may be $m(\lambda)$, the *multiplicity* of λ, defined as $\dim E(\lambda)$ and $\dim E_g(\lambda)$, the *generalized multiplicity* of λ, denoted $m_g(\lambda)$. Obviously $E(\lambda) \subseteq E_g(\lambda)$, so that $m(\lambda) \leq m_g(\lambda)$.

3.9 Example. *Spectral theory of finite-dimensional operators.* Let $E = \mathbb{C}^m$ ($\mathbb{C}^m$ is m-dimensional unitary space), A an operator from E into E. Then (a) $\sigma(A)$ consists of a finite number of eigenvalues $\lambda_1, \ldots, \lambda_n$ $(n \leq m)$, (b) $E = E_g(\lambda_1) \oplus \cdots \oplus E_g(\lambda_n)$, where $\oplus$ indicates direct sum (so that $m_g(\lambda_1) + \cdots + m_g(\lambda_n) = m$).

***3.10 Example.** *Spectral theory of compact operators.* An everywhere defined operator A in a Banach space E is *compact* if and only if $\{Au_n\}$ contains a convergent subsequence whenever $\{u_n\}$ is bounded. Then (a) $\sigma(A)$ consists of a (finite or countable) sequence $\{\lambda_1, \lambda_2, \ldots\}$ of complex numbers having an accumulation point at zero when the sequence is infinite, and (b) Each nonzero $\lambda \in \sigma(A)$ is an eigenvalue with $m_g(\lambda) < \infty$ (hence with finite multiplicity). However, unlike in the finite-dimensional case, the space E may not be spanned by the generalized eigenvectors, even in an approximate sense. Consider the *Volterra operator*

$$Vu(x) = \int_0^x u(t)\, dt \tag{3.12}$$

in $L^2(0,1)$; A is compact and $\sigma(A) = \{0\}$. Since 0 is not an eigenvalue of V, V has no generalized eigenvectors at all (see Dunford-Schwartz [1958: 1, Sec. 7.4] or Kato [1976: 1, Sec. 3.7] for complete treatments of spectral properties of compact operators).