Product Integration
with Applications to Differential Equations

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ENCYCLOPEDIA OF MATHEMATICS and Its Applications

GIAN-CARLO ROTA, *Editor*
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
Volume 10

Section: Analysis
Felix E. Browder, *Section Editor*

Product Integration with Applications to Differential Equations

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1979

Addison-Wesley Publishing Company
Advanced Book Program
Reading, Massachusetts

London•Amsterdam•Don Mills, Ontario•Sydney•Tokyo

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521302302

First published 1979 by Addison Wesley
First published by Cambridge University Press 1984

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Dollard, John Day.
Product integration with applications to
differential equations.

(Encyclopedia of mathematics and its applica-
tions ; v. 10)
Bibliography: p.
Includes index.
1. Differential equations. 2. Differential
equations, Partial. 3. Integral equations.
I. Friedman, Charles N., joint author. II. Title.
III. Series.
QA371.D64 515′.35 79-20454

ISBN 978-0-521-30230-2 Hardback

Cambridge University Press
978-0-521-30230-2 - Product Integration with Applications to Differential Equations
John D. Dollard and Charles N. Friedman
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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

This is the first survey of the product integral since the turn of the century. Product integration, an idea going back to Volterra, has been and is being used today in several disparate circumstances; the present definitive treatment will contribute to make better known this useful technique.

P. R. Masani, in his Appendix, gives a different overview of the history of the subject. Professor Masani has adopted the colorful notation of the first workers. His subject is largely complementary to the main text. His different notation for the product integral will be useful to those who wish to gain access to the early literature.

GIAN-CARLO ROTA

Foreword

An editor's preface to a mathematics book does not have a clearly defined role in contemporary usage. If some past precedents were followed, I would merely remark that the present work by John Dollard and Charles Friedman is a completely self-contained treatment of the product integral on a simple and elementary basis. As such, I believe it to be unique as far as this topic is concerned. The applications that are presented fall mainly within the domain of ordinary differential equations. Some amplifications of the generality of the theme with applications to a wider circle of mathematical topics are described in the accompanying Appendix by P. R. Masani.

For the benefit of some readers at least, I shall go beyond this conventional restriction of the editor's function. Though in the last analysis, mathematical topics must be treated in full technical detail and with logical completeness (as they are indeed treated in the body of the present work), it is often useful to preface such a detailed development with a more discursive and less technical discussion.

What is the product integral? As the text tells us, it is an analytic process or class of processes first put forward by Volterra in the last decades of the nineteenth century for the study of various questions relating to the theory of ordinary differential equations. As Professor Masani reminds us in his Appendix, it was extensively developed by Schlesinger in the early part of the twentieth century, particularly in connection with differential equations in the complex domain. In later decades, it was investigated in connection with topics in functional analysis, stochastic processes, and the theory of analytic functions with operator values. It is based on the same heuristic idea which in the domain of quantum field theory was used by Feynman with such remarkable impact in his celebrated path-integral approach.

What is this heuristic idea? To describe it, let us form a simple model of the definition of a product integral. Consider a metric space M and a class $T(M)$ of self-mappings of M. Assume that this class is closed under compositions. Suppose that we are given a mapping $S(t, h)$ in this class $T(M)$ for each real parameter t in a given interval $[a, b]$ of the real line and for each real h with $0 < h < d$. For each partition of the interval $[a, b]$ with

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

such that each successive difference $t_{j+1} - t_j < d$, we can form the iterated composition of the mappings

$$S(t_{n-1}, t_n - t_{n-1}) \cdots S(t_1, t_2 - t_1)S(t_0, t_1 - t_0)$$

(where we note that the factors have to be composed in the prescribed *time-ordered* fashion). If, as the mesh of the partition approaches zero, these iterated composition mappings converge to a limit mapping T in a prescribed topology of convergence in $T(M)$, then we call this limit T the product integral, written in the form

$$T = \prod_a^b S(t, dt).$$

We have thus defined a Riemann product integral, with possible extensions to corresponding more-general product integrals of the Lebesgue type.

Where do such iterated compositions and their limits arise in a natural way? First and foremost, in the theory of ordinary differential equations in both the finite- and infinite-dimensional cases. Consider a differential equation of first order on a manifold M,

$$\frac{du}{dt} = A(t, u(t)).$$

Suppose that for a solution $u(t)$ of this equation on an interval $[a, b]$ under suitable conditions on the equation, for $s < t$, $u(s)$ determines $u(t)$. Then set

$$T(s, t)(u(s)) = u(t)$$

thereby defining the two-parameter family of mappings $T(s, t)$ called the *propagator* mappings corresponding to the given equation. Suppose that in addition we are given in some natural way a nice class of *pseudo-propagators* $S(t, h)$, i.e., mappings of M into M which satisfy the differential equation

$$\frac{d}{dh} S(t, h)(u)|_{h=0} = A(t, u).$$

One can form the product integral

$$W_{a,b} = \prod_a^b S(t, dt)$$

and can hope that $w_{a,b} = T(a, b)$ for all such pairs (a, b). In a wide variety of cases, this procedure yields useful machinery for constructing and studying the solutions of differential equations.

The simplest such case is that in which M reduces to a vector space V (as it always does locally) and we set

$$S(t, h)(u) = u + hA(t, u).$$

In this particularly simple case, the iterated composition of these particular pseudo-propagators corresponds simply to the application of the classical Euler polygon method for solving an initial-value problem for the ordinary differential equation. In other words, we approximate the differential equation by the system of difference equations

$$u(t_{j+1}) - u(t_j) = (t_{j+1} - t_j)A(t_j, u(t_j)),$$

and take the limit as the mesh of the partition goes to zero. If this approach works, we can write

$$T(s, r) = \prod_s^r (I + dt\ A(t)).$$

A more sophisticated case appears when the operators $A(t)$ are not everywhere defined and therefore cannot be applied to an arbitrary element of V, a situation characteristic of ordinary differential equations in infinite-dimensional spaces V obtained from initial-value problems for partial differential equations. In this case, we modify the difference equation of the preceding paragraph to obtain instead

$$u(t_{j+1}) - u(t_j) = (t_{j+1} - t_j)A(t_j, u(t_{j+1})).$$

If we solve this latter system, we find that

$$u(t_{j+1}) = (I - (t_{j+1} - t_j)A(t_j))^{-1}(u(t_j)).$$

The successive application of this formula yields an iterated composition of mappings defining the approximation for the product integral corresponding to $S(t, h) = (I - hA(t))^{-1}$. Where this approximation works, we find that

$$T(s, r) = \prod_s^r (I - dtA(t))^{-1}.$$

A classical case for linear operators in a Banach space is that treated in the Hille-Yosida theorem in which $A(t)$ is a fixed closed linear operator A satisfying the condition that

$$\|(I - hA)^{-1}\| \leq 1.$$

Here the product integral yields the semigroup generated by A.

A third illuminating special case involves solving the differential equation

$$\frac{du}{dt}(t) = A(t, u(t)) + B(t, u(t))$$

by what is called the fractional-step method in numerical analysis. Suppose that for each j we define

$$v_j - u(t_j) = (t_{j+1} - t_j)A(t_j, v_j)$$
$$u(t_{j+1}) - v_j = (t_{j+1} - t_j)B(t_j, u(t_{j+1})).$$

If we add the two formulas, we find a reasonable looking approximation to our differential equation, namely,

$$u(t_{j+1}) - u(t_j) = (t_{j+1} - t_j)[A(t_j, v_j) + B(t_j, u_{j+1})].$$

This suggests using the product integral for

$$S(t, h) = (I - hB(t))^{-1}(I - hA(t))^{-1}.$$

The success of this process in various cases leads to the representation of propagators for such equations by Trotter formulas.

The reader should bear in mind some general facts about product integrals.

Product integration is in fact a generalization in principle of the ordinary additive process of integration if one interprets the latter as applying to commuting operator-valued functions. If V is a normed linear space and if $A(t)$ is a continuous family of bounded linear operators on V with $A(t)$ and $A(s)$ commuting for each pair s and t, then

$$\prod_r^s \exp(dt\, A(t)) = \exp\left(\int_r^s A(t)\, dt\right)$$

where the exponential function is defined in the simplest way by the infinite power series, $\exp(T) = \Sigma_{j=0}^{\infty} T^j/j!$. However, product integration is fundamentally *noncommutative*. Since transformations (even linear ones) generally do not commute, the definition of the product integrals depends essentially on the *time-ordered* structure of the integrand.

Product integration is not a linear operation, nor in general is it restricted to linear operators or transformations. The linear case is often simpler in its structure, but the basic machinery is also applicable to the nonlinear case with appropriate modifications.

Finite dimensionality is also not a basic requirement for the application of product integrals. Indeed, the second illustrative case mentioned above is significant mainly in the context of operators defined in infinite-dimensional spaces. Even if we begin with a finite-dimensional context and then introduce such basic kinds of additional structure as random processes, we are led immediately and naturally into an infinite-dimensional situation. (For a formal account, we refer to Professor Masani's Appendix.)

When Volterra first defined product integrals in the late 1880s, the definition was part of the same movement to formulate analytical processes in explicitly self-conscious and general terms that led to the development of the beginnings of twentieth-century functional analysis—a process in which Volterra was one of the most explicit public spokesmen as shown by his address to the Paris International Congress of Mathematicians in 1900. Though overshadowed by more fashionable parts of analysis and functional analysis, the theory of the product integral in its various forms continues to be a focus of important activity. This is particularly the case in the theory of nonlinear evolution equations and their extension to nonstationary evolution processes. In the past decade and a half, iteration processes of the product integral type have been adapted to the study of a general theory of nonlinear semigroups, initiated in Hilbert space by Komura and extensively developed in a more general Banach space context as a nonlinear analogue of the older Hille-Yosida theory of linear semigroups.

Professors Dollard and Friedman have worked primarily on mathematical questions originating in mathematical physics. In the present book they have given the reader a concrete, simply written presentation of a basic tool in present-day analysis which should be of value to applied mathematicians and mathematical physicists as well as to mathematical analysts.

FELIX E. BROWDER
General Editor, Section on Analysis

Preface

This monograph is intended as an introduction to product integration for the general scientific audience and a reference book for workers in differential equations. Chapter 1 can be understood by a reader with only a knowledge of matrix algebra and elementary calculus. Later chapters assume additional knowledge on the part of the reader.

In a nutshell, the product integral is to the product what the ordinary integral is to the sum. The product integral arises in connection with *equations of evolution* and the associated initial-value problems. These have the form

$$y'(x) = A(x)y(x) \qquad y(x_0) = I \tag{1}$$

where y and A are operator-valued functions (in the simplest case they are nxn matrix-valued functions), and I is the identity operator. Many important scientific equations can be analyzed by solving initial-value problems of the type given in (1). Examples are the heat equation, the Schrödinger equation, and any linear ordinary differential equation. The product integral is a construction which solves the initial-value problem (1). Analogously, the ordinary integral $\int_{x_0}^{x} A(s)ds$ is a construction which solves the initial-value problem

$$y'(x) = A(x) \qquad y(x_0) = 0 \tag{2}$$

It is well known that there are many advantages to be gained by studying the ordinary integral as an object in its own right, rather than restricting oneself to the terminology "the solution of (2)." There are similar advantages to be gained by studying the product integral as an object in its own right. Strangely, this is usually not done, and many scientists are unfamiliar with the concept of product integration. The present authors attribute this to the fact that no comprehensive modern treatise on product integration has been available. V. Volterra, who invented product integration, wrote a monograph with B. Hostinski in 1938 [VV4], which deals with the fundamentals of the subject. However, their account is limited in generality by modern standards, and is difficult to read because of unnatural notational conventions and an approach which obscures the simplicity of the subject. As a consequence, those wishing to exploit product integral ideas have tended to reinvent the subject for themselves or rely on scanty accounts in research papers. It is a testimony to the usefulness of product integral concepts that advanced workers in differential equations have learned and exploited these concepts despite the unavailability of a standard text on the subject. The present authors think that the time is ripe for a monograph which will introduce product integrals to the general scientific audience.

In this monograph we deal almost exclusively with "linear" product integration, which is the theory obtained when $A(x)$ of eq. (1) is a linear operator for each x. This is the only case for which a systematic theory can be said to exist. The literature contains a number of interesting and important applications to cases in which $A(x)$ is nonlinear. However, the existence theory for such cases typically depends on very specialized hypotheses and cannot be viewed as resulting from a general theory in a natural way. For this reason, an account of this work would consist in little more than a recital of the contents of the relevant papers, and we chose to give only some brief remarks on this work along with references to the literature. We believe that our account of the linear case will serve as a sufficient introduction to the literature on nonlinear product integration.

Our account of the theory of linear product integration is quite extensive. Beginning with the case in which $A(x)$ is an $n \times n$ matrix, we point out the beautiful simplifications brought to the theory of linear ordinary differential equations by viewing them from the product integral viewpoint. Building on the foundation laid in the discussion of $n \times n$ matrices, we arrive by successive generalizations at an advanced theory suitable for the study of partial differential equations. (In this theory $A(x)$ could be, say, an unbounded operator on a Hilbert space.) Enough applications are given to illustrate the usefulness and flexibility of the theory.

JOHN D. DOLLARD
CHARLES N. FRIEDMAN

Introduction

We indicate here, briefly, the content of the chapters. Chapter 1 is an elementary introduction to the product integral of a continuous matrix-valued function and its properties. (The generalization to "Lebesgue product integrals" is sketched in Section 8.) This chapter should be accessible to readers with quite minimal mathematical background. It is prerequisite for understanding of the other chapters. Chapter 2 deals with contour product integrals; the development is parallel to that of the theory of ordinary contour integrals in complex variable theory. Contour product integrals are not used in later chapters. In Chapter 3 we present a theory of product integration in a much more general setting. More mathematical background is required here, for example, familiarity with functional analysis in Banach space and some integration theory. Included are results on the equation of evolution (1) with unbounded $A(x)$. The other chapters (except Chapter 4, Section 5) are independent of this chapter. Chapter 4 presents applications of product integration to the theory of differential equations; some new results concerning solutions of the Schrödinger equation with rather singular potentials are included. In Chapter 5 we discuss product integration of (matrix-valued) measures. Some familiarity with measure theory is assumed here, but nothing very sophisticated is required. Chapter 6 contains a discussion of work on product integration by various other authors and some remarks on generalizations of the theory. Following Chapter 6 is an appendix on matrix theory containing elementary definitions and results and a few special results which may not be familiar to all readers. We have included, finally, a fairly extensive and hopefully complete list of references together with notes. This includes all papers, articles, books, etc., known to us, which develop, discuss, or make use of product integration.

In each chapter, theorems, definitions, and formulas are numbered consecutively within each section. Thus within Chapter 1, definition 1.2 refers to the second definition of Section 1, the notation (2.3) refers to the third formula of Section 2, etc. References in a chapter to theorems (definitions, etc.), in a previous chapter are given by stating the theorem number and chapter, e.g., "Theorem 5.1 of Chapter 2" or by prefixing an Arabic numeral indicating the chapter; e.g., the theorem just cited would be referred to as "Theorem 2.5.1."

The authors would like to thank Professors M. Crandall, J. R. Dorroh,

W. E. Fitzgibbon, J. Goldstein, J. Helton, J. S. MacNerney, P. R. Masani, J. Neuberger, R. Showalter, B. Simon, M. K. Smith, and G. Webb for suggestions and help in compiling the list of references. Thanks are also due to B. Simon for several interesting discussions on product integration and applications of the theory.

The authors are also pleased to acknowledge a grant of one thousand dollars from the University Research Institute of the University of Texas at Austin.

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