CHAPTER 1

Product Integration of Matrix-Valued Functions

1.0 Introduction

In this chapter we define product integration for $n \times n$ matrix-valued functions. As will be seen, the concept of product integration can be used as a central unifying idea in the study of systems of linear differential or integral equations. A large portion of the present chapter is devoted to an elaboration of this point. We begin with a brief explanation of the intuitive connection between linear differential equations and product integrals. (For the definition of various standard terms in matrix theory, see Appendix 1.)

Consider a system of n linear differential equations in n unknowns, having the form

$$\begin{aligned} y_1'(s) &= a_{11}(s)y_1(s) + a_{12}(s)y_2(s) + \cdots + a_{1n}(s)y_n(s) \\ &\vdots \\ y_n'(s) &= a_{n1}(s)y_1(s) + a_{n2}(s)y_2(s) + \cdots + a_{nn}(s)y_n(s). \end{aligned} \tag{0.1}$$

The coefficients $a_{ij}(s)$ are assumed to be continuous on an interval $[a,b]$. At the endpoints of the interval, the derivatives are one-sided. Suppose that the values $y_1(a), y_2(a), \ldots, y_n(a)$ are given. Consider the problem of finding $y_1(b), y_2(b), \ldots, y_n(b)$. We first convert to matrix notation: writing

$$y(s) = \begin{bmatrix} y_1(s) \\ \vdots \\ y_n(s) \end{bmatrix} \tag{0.2}$$

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and

$$A(s)=\begin{bmatrix} a_{11}(s) & \cdots & a_{1n}(s) \\ \vdots & & \vdots \\ a_{n1}(s) & \cdots & a_{nn}(s) \end{bmatrix}, \tag{0.3}$$

Eq. (0.1) take on the form

$$Y'(s)=A(s)Y(s), \tag{0.4}$$

and the problem is to calculate $Y(b)$ given $Y(a)$. An approximate value for $Y(b)$ can be found by a variant of the Euler tangent-line method: let $P=\{s_0,s_1,\ldots,s_n\}$ be a partition of the interval $[a,b]$. Let $\Delta s_k=s_k-s_{k-1}$ for $k=1,\ldots,n$. On the interval $[s_0,s_1]$ we approximate $A(s)$ by the constant value $A(s_1)$. We solve the differential equation (0.4) on the interval $[s_0,s_1]$ with $A(s)$ replaced by $A(s_1)$ and with initial value $Y(a)$. The solution is $e^{A(s_1)(s-a)}Y(a)$. This leads to the following approximate value for the exact solution Y at the point s_1:

$$Y(s_1)\cong e^{A(s_1)(s_1-a)}Y(a)=e^{A(s_1)\Delta s_1}Y(a). \tag{0.5}$$

Now proceeding on the interval $[s_1,s_2]$ from the approximate initial value (0.5) and replacing $A(s)$ by the constant value $A(s_2)$, we find

$$Y(s_2)\cong e^{A(s_2)\Delta s_2}e^{A(s_1)\Delta s_1}Y(a). \tag{0.6}$$

Proceeding in this manner, the following approximate value is obtained for $Y(b)$:

$$Y(b)=Y(s_n)\cong e^{A(s_n)\Delta s_n}\cdots e^{A(s_1)\Delta s_1}Y(a)=\prod_{k=1}^{n} e^{A(s_k)\Delta s_k}Y(a). \tag{0.7}$$

We remark that on the right-hand side of Eq. (0.7) the order of the exponentials is important: they may not commute, because in general the values $A(s_1),A(s_2),\ldots,A(s_n)$ will not commute. The ordered product of exponentials in (0.7) will be denoted $\Pi_P(A)$:

$$\Pi_P(A)=\prod_{k=1}^{n} e^{A(s_k)\Delta s_k}. \tag{0.8}$$

Let $\mu(P)$ denote the mesh of the partition P (length of the longest subinterval). The function A is continuous on $[a,b]$, hence uniformly continuous. Thus if $\mu(P)$ is small, for all $k=1,\ldots,n$, the value $A(s_k)$ will be close to the values taken by A on the interval $[s_{k-1},s_k]$. It is thus

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reasonable to suppose that when $\mu(P)$ is small the calculation given above produces a value for $Y(b)$ quite close to the true value. We may then expect to find $Y(b)$ exactly by the prescription

$$Y(b) = \lim_{\mu(P)\to 0} \Pi_P(A)\,Y(a). \tag{0.9}$$

This is indeed correct, as will be seen. In fact, the limit of the matrix $\Pi_P(A)$ exists independent of the initial value $Y(a)$, and it is the limit of $\Pi_P(A)$ that defines the product integral of A over $[a,b]$.

We will in the sequel be mainly concerned with *analytical* properties of differential equations and product integrals. However, it is important to note that the construction of the product integral, as outlined above, is very close in spirit to procedures for finding *numerical* solutions to differential equations. In fact, the method suggested above for finding $Y(b)$ is a possible (although by no means the most efficient) method for beginning a numerical calculation of $Y(b)$. By taking a sufficiently fine partition and making sufficiently good approximations of the exponentials occurring, the value of $Y(b)$ can be calculated to any desired degree of accuracy. It is this connection with the ideas of numerical analysis that gives the subject of product integration its very *constructive* flavor. In this subject, we seek not only to prove *existence* of solutions of differential equations, but to represent these solutions concretely as limits of certain approximate solutions. Many of the theorems of this chapter concerning existence of the product integral can be reinterpreted as theorems concerning the accuracy of certain numerical approximations to the solutions of differential equations. The reader who bears in mind the connection with numerical analysis will have a deeper understanding of the subject.

We now turn to the study of the product integral. In the course of our study, we will justify the analysis of Eq. (0.4) given above.

1.1 Product Integration

The ingredients of the approximation procedure discussed above are (i) a step-function, namely, the step-function taking the value $A(s_1)$ on $[s_0,s_1]$, $A(s_2)$ on $(s_1,s_2]$, etc., and (ii) the construction of the corresponding ordered product (0.8) used in approximating $Y(b)$.

We now formalize our construction, extending it to any step-function and any $x\in[a,b]$.

DEFINITION 1.1. *Let B be a function defined on the interval $[a,b]$ of real numbers and taking values in the space $\mathbb{C}_{n\times n}$ of $n\times n$ matrices with complex entries. B is called a step-function if and only if there is a partition $P=\{s_0,s_1,\ldots,s_n\}$ of $[a,b]$ such that B is constant on each open subinterval (s_{k-1},s_k) for $k=1,\ldots,n$. The value of B on (s_{k-1},s_k) is then denoted B_k.*

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DEFINITION 1.2. *Let* $A:[a,b]\to\mathbb{C}_{n\times n}$ *be a function, and let* $P=\{s_0,s_1,\ldots,s_n\}$ *be a partition of* $[a,b]$. *The* point-value approximant A_P *corresponding to* A *and* P *is the step-function taking value* $A(s_1)$ *on* $[s_0,s_1]$, $A(s_2)$ *on* $(s_1,s_2],\ldots,$ *and* $A(s_n)$ *on* $(s_{n-1},s_n]$.

DEFINITION 1.3. *Let* $B:[a,b]\to\mathbb{C}_{n\times n}$ *be a step-function. Using notation as in Definition* 1.1, *and letting* $\Delta s_k=s_k-s_{k-1}$, *the function* $E_B:[a,b]\to\mathbb{C}_{n\times n}$ *is defined as follows*:

$$\begin{aligned} E_B(x)&=e^{B_1(x-s_0)}, && x\in[s_0,s_1]\\ &=e^{B_2(x-s_1)}e^{B_1\Delta s_1}, && x\in[s_1,s_2] \qquad (1.1)\\ &\ \vdots\\ &=e^{B_n(x-s_{n-1})}\cdots e^{B_2\Delta s_2}e^{B_1\Delta s_1}, && x\in[s_{n-1},s_n]. \end{aligned}$$

We can now describe the approximation procedure of the Introduction as follows: Given the continuous function A and a partition P of $[a,b]$, we formed the point-value approximant A_P and constructed the ordered product $\Pi_P(A)$ of Eq. (0.8), which is precisely $E_{A_P}(b)$. The conclusion was that $E_{A_P}(b)Y(a)$ should be close to $Y(b)$ if the mesh of P is small. In view of the definition of $E_{A_P}(x)$, it is equally reasonable to conclude that $E_{A_P}(x)Y(a)$ should be close to $Y(x)$ when the mesh of P is small. This will follow from our analysis.

Let A be continuous (hence uniformly continuous) on $[a,b]$. Taking $\varepsilon>0$, choose $\delta>0$ such that $x,y\in[a,b]$ and $|x-y|<\delta$ imply $\|A(x)-A(y)\|<\varepsilon$. It follows immediately that for $\mu(P)<\delta$ we have

$$\|A_P(x)-A(x)\|<\varepsilon \qquad \text{for all} \qquad x\in[a,b]. \tag{1.2}$$

Thus we have uniformly on $[a,b]$

$$\lim_{\mu(P)\to 0} A_P(x)=A(x). \tag{1.3}$$

It follows immediately that the step-function A_P converges *in the* L^1 *sense* to A as $\mu(P)\to 0$. This means the following: If B is an $n\times n$ matrix-valued function on $[a,b]$, all of whose entries are integrable over $[a,b]$, we set

$$\|B\|_1=\int_a^b\|B(s)\|\,ds. \tag{1.4}$$

To say that A_P converges to A in the L^1 sense means that the following equation holds:

$$\lim_{\mu(P)\to 0}\|A_P-A\|_1=0. \tag{1.5}$$

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This will be the crucial property of the point-value approximants in justifying our approximation procedure for the differential equation (0.4). The fact that A_P converges to A uniformly is of interest only in deducing Eq. (1.5). In fact, our approximation procedure could equally well be carried out using *any* sequence of step-functions converging to A in the L^1 sense. This fact will be reflected in Theorems 1.1 and 1.2, which justify the approximation procedure. Before proceeding to Theorem 1.1, we prove two simple lemmas on the properties of the functions $E_B(x)$ of Definition 1.3.

LEMMA 1.1. *Let* $B:[a,b]\to\mathbb{C}_{n\times n}$ *be a step-function and let* E_B *be as in Definition* 1.3. *Then*

(*i*) $E_B(a)=I$ (*I denotes the $n\times n$ identity matrix*).
(*ii*) $E_B(x)$ *is nonsingular for each* $x\in[a,b]$.
(*iii*) E_B *and* E_B^{-1} *are continuous on* $[a,b]$ *and satisfy the integral equations*

$$E_B(x)=I+\int_a^x B(s)E_B(s)\,ds \tag{1.6}$$

and

$$E_B^{-1}(x)=I-\int_a^x E_B^{-1}(s)B(s)\,ds. \tag{1.7}$$

(*iv*) E_B *and* E_B^{-1} *satisfy the bounds*

$$\begin{aligned} \|E_B(x)\| &\leqslant e^{\int_a^x \|B(s)\|\,ds}, \\ \|E_B^{-1}(x)\| &\leqslant e^{\int_a^x \|B(s)\|\,ds}. \end{aligned} \tag{1.8}$$

Proof. Parts (i) and (ii) are obvious. Note that $E_B^{-1}(x)$ is obtained by reversing the order of the factors in $E_B(x)$ and inserting minus signs in the exponents. E_B and E_B^{-1} are continuous by inspection. Also, if P is the partition associated with B, then inspection of the definition (1.1) shows that E_B is differentiable except at the division points of P, and

$$E_B'(x)=B(x)E_B(x) \tag{1.9}$$

except at the division points. Since $E_B(a)=I$, the integral equation (1.6) is essentially just the fundamental theorem of calculus applied to E_B. Some additional justification is necessary because E_B is not differentiable at all points of $[a,b]$; however, E_B is continuous on $[a,b]$ and is continuously differentiable on each open subinterval (s_{k-1},s_k) determined by P. Using the fundamental theorem of calculus on each open subinterval and piecing the results together, one easily obtains Eq. (1.6). Equation (1.7) is obtained in a similar manner. Thus (iii) is proved. To prove part (iv), we adopt the

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notation of Definition 1.3. Letting $x\in[s_{k-1},s_k]$, we have

$$\begin{aligned}\|E_B(x)\| &= \|e^{B_k(x-s_{k-1})}\cdots e^{B_1\Delta s_1}\|\\ &\leqslant \|e^{B_k(x-s_{k-1})}\|\cdots\|e^{B_1\Delta s_1}\|\\ &\leqslant e^{\|B_k\|(x-s_{k-1})}\cdots e^{\|B_1\|\Delta s_1}\\ &= e^{\|B_k\|(x-s_{k-1})+\cdots+\|B_1\|\Delta s_1} = e^{\int_a^x\|B(s)\|\,ds}. \end{aligned}\tag{1.10}$$

The corresponding bound for E_B^{-1} is proved in an analogous manner. Thus the lemma is proved. ■

LEMMA 1.2. *Let $B,C:[a,b]\to\mathbb{C}_{n\times n}$ be step-functions. Then*

$$E_B(x)-E_C(x)=E_C(x)\int_a^x E_C^{-1}(s)(B(s)-C(s))E_B(s)\,ds. \tag{1.11}$$

Proof. The proof of (1.11) is very similar to the proof of Eq. (1.6) above. Let

$$G(x)=E_C^{-1}(x)E_B(x). \tag{1.12}$$

Then $G(a)$ is the identity, G is continuous, and, except at the division points of the partitions associated with B and C, G is differentiable. The product rule for differentiation and the definitions of E_B and E_C^{-1} show that except at the division points we have

$$G'(x)=E_C^{-1}(x)(B(x)-C(x))E_B(x). \tag{1.13}$$

Thus, arguing as in the proof of the integral equation (1.6), we have

$$G(x)=I+\int_a^x E_C^{-1}(s)(B(s)-C(s))E_B(s)\,ds. \tag{1.14}$$

Equation (1.11) is obtained from (1.14) by multiplying on the left with $E_C(x)$. This proves the lemma. ■

We are now in a position to prove the following:

THEOREM 1.1. *Let $A:[a,b]\to\mathbb{C}_{n\times n}$ be continuous and let $\{A_n\}$ be any sequence of step-functions converging to A in the L^1 sense. Then the sequence $\{E_{A_n}(x)\}$ converges uniformly on $[a,b]$ to a matrix denoted $\prod_a^x e^{A(s)ds}$, called the* product integral of A over $[a,x]$.

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Proof. By Lemma 1.2, we have

$$E_{A_n}(x)-E_{A_m}(x)=E_{A_m}^{-1}(x)\int_a^x E_{A_m}(s)(A_n(s)-A_m(s))E_{A_n}(s)\,ds. \tag{1.15}$$

Thus

$$\|E_{A_n}(x)-E_{A_m}(x)\|\leqslant\|E_{A_m}^{-1}(x)\|\int_a^x\|E_{A_m}(s)\|\,\|A_n(s)-A_m(s)\|\,\|E_{A_n}(s)\|\,ds. \tag{1.16}$$

The norms of the functions $E_{A_m}^{-1}$, E_{A_m}, and E_{A_n} can be estimated using (1.8), and we may estimate the integrals in (1.8) by integrals from a to b. Using also the notation (1.4), we then obtain from (1.16) the estimate

$$\begin{aligned}\|E_{A_n}(x)-E_{A_m}(x)\|&\leqslant e^{2\|A_m\|_1}e^{\|A_n\|_1}\int_a^x\|A_n(s)-A_m(s)\|\,ds\\&\leqslant e^{2\|A_m\|_1}e^{\|A_n\|_1}\|A_n-A_m\|_1.\end{aligned} \tag{1.17}$$

Now because $\{A_n\}$ converges to A in the L^1 sense, the norms $\|A_n\|_1$ are bounded and the sequence $\{A_n\}$ is Cauchy in the L^1 sense. Thus the right-hand side of (1.17) approaches zero as $n,m\to\infty$. Furthermore, the right-hand side of (1.17) is independent of x. This shows that the sequence $\{E_{A_n}(x)\}$ is uniformly Cauchy, hence uniformly convergent. If $\{B_n\}$ and $\{C_n\}$ are two sequences converging to A in the L^1 sense, then estimating $\|E_{B_n}(x)-E_{C_n}(x)\|$ as above, one sees that $\{E_{B_n}\}$ and $\{E_{C_n}\}$ have the same limit. Thus the theorem is proved. We note for later reference that since $E_{A_n}(a)=I$ for all n, we have $\prod_a^a e^{A(s)\,ds}=I$. ■

It should be noted that in the proof of Theorem 1.1 no use was made of the fact that A was continuous. All that was needed was a sequence of step-functions converging to A in the L^1 sense. This fact permits the definition of the product integral for any matrix function A with Lebesgue integrable entries, as we shall remark later. However, for the present we continue to develop the theory under the hypothesis that A is continuous. This is the most natural assumption when studying the differential equation (0.4).

Since we know in particular that for continuous A there is a sequence of point-value approximants converging to A in the L^1 sense, Theorem 1.1 implies that in our previous notation

$$\prod_a^b e^{A(s)\,ds}=\lim_{\mu(P)\to 0}\prod_{k=1}^n e^{A(s_k)\Delta s_k}, \tag{1.18}$$

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where $P=\{s_0,s_1,\ldots,s_n\}$ denotes a partition of $[a,b]$. Naturally the product integral $\prod_a^x e^{A(s)ds}$ can be evaluated in an analogous manner using partitions P of the interval $[a,x]$.

We can make a slight generalization of (1.18), which is occasionally convenient. Namely, given a partition P, instead of considering the point-value approximant to A we can consider a step-function that, on the subinterval $(s_{k-1},s_k]$ of P, takes the value $A(s_k')$, where s_k' is any number in $[s_{k-1},s_k]$. When $\mu(P)\to 0$ such a step-function will still approach A in the L^1 sense (even uniformly) and we therefore have

$$\prod_a^b e^{A(s)ds}=\lim_{\mu(P)\to 0}\prod_{k=1}^n e^{A(s_k')\Delta s_k}. \tag{1.18$'$}$$

We now establish some basic properties of the product integral.

THEOREM 1.2. *Let $A:[a,b]\to\mathbb{C}_{n\times n}$ be continuous, and for $x\in[a,b]$ let*

$$F(x,a)=\prod_a^x e^{A(s)ds}. \tag{1.19}$$

Then F satisfies the integral equation

$$F(x,a)=I+\int_a^x A(s)F(s,a)\,ds. \tag{1.20}$$

The function F is also a solution of the initial value problem

$$\frac{dF(x,a)}{dx}=A(x)F(x,a),\qquad F(a,a)=I \tag{1.21}$$

for $x\in[a,b]$. (One-sided derivatives are meant at a and b.)

Proof. Let $\{A_n\}$ be a sequence of step-functions converging to A in the L^1 sense. The functions E_{A_n} are continuous on $[a,b]$ and converge uniformly to F as $n\to\infty$. Thus $F(x,a)$ is continuous. Now by Lemma 1.1 we have

$$E_{A_n}(x)=I+\int_a^x A_n(s)E_{A_n}(s)\,ds. \tag{1.22}$$

Taking the limit of (1.22) as $n\to\infty$, and using convergence of A_n in the L^1 sense and uniform convergence of E_{A_n}, we easily obtain Eq. (1.20). Then because the integrand in (1.20) is continuous, the differential equation in (1.21) follows immediately from (1.20). That $F(a,a)=I$ was remarked earlier. This completes the proof. ■

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We note that according to this proof the product integral of a continuous function is continuously differentiable.

Theorem 1.2 provides the justification for the approximation procedure discussed earlier for Eq. (0.4), because it is now clear that if $F(x,a)$ is defined as in (1.19), then the function $F(s,a)Y(a)$ satisfies (0.4) and reduces to $Y(a)$ when $x=a$. As is well known and as we shall soon prove, there is only one function satisfying these conditions, hence $Y(s)=F(s,a)Y(a)$ for all s, and in particular for $s=b$, the case studied in the approximation procedure.

It is useful to remark that the product integral of a commutative family can be found explicitly:

THEOREM 1.3. *Let $A:[a,b]\to\mathbb{C}_{n\times n}$ be continuous, and suppose that the family $\{A(s)|s\in[a,b]\}$ is* commutative, *i.e.*,

$$A(s)A(s')=A(s')A(s) \qquad \text{for all } s,s' \text{ in } [a,b]. \tag{1.23}$$

Then

$$\prod_a^x e^{A(s)\,ds}=e^{\int_a^x A(s)\,ds}. \tag{1.24}$$

Proof. Letting P denote a partition of $[a,x]$, we have

$$\begin{aligned}\prod_a^x e^{A(s)\,ds} &= \lim_{\mu(P)\to 0}\prod_{k=1}^n e^{A(s_k)\Delta s_k}\\ &= \lim_{\mu(P)\to 0} e^{\sum_{k=1}^n A(s_k)\Delta s_k},\end{aligned} \tag{1.25}$$

where we have used commutativity to obtain the second equality. But $\sum_{k=1}^n A(s_k)\Delta s_k$ is just a Riemann sum for the integral $\int_a^x A(s)\,ds$, so the right-hand side of (1.25) agrees with that of (1.24). Thus the theorem is proved. ■

We now return to the development of the general theory. The next theorem establishes a fundamental property of product integrals.

THEOREM 1.4. *Let $A:[a,b]\to\mathbb{C}_{n\times n}$ be continuous. Then for each $x\in[a,b]$ the product integral $\prod_a^x e^{A(s)\,ds}$ is nonsingular. The following formula holds:*

$$\det\left(\prod_a^x e^{A(s)\,ds}\right)=e^{\int_a^x \operatorname{tr}A(s)\,ds}\neq 0. \tag{1.26}$$

Proof. Clearly if (1.26) is established, then nonsingularity of the product integral follows. To establish (1.26), note that if P denotes a partition of

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$[a,x]$, then

$$\det\left(\prod_a^x e^{A(s)\,ds}\right)=\det\left(\lim_{\mu(P)\to 0}\prod_{k=1}^n e^{A(s_k)\Delta s_k}\right)$$
$$=\lim_{\mu(P)\to 0}\det\left(\prod_{k=1}^n e^{A(s_k)\Delta s_k}\right)=\lim_{\mu(P)\to 0}\prod_{k=1}^n \det\left(e^{A(s_k)\Delta s_k}\right)$$
$$=\lim_{\mu(P)\to 0}\prod_{k=1}^n e^{\operatorname{tr}A(s_k)\Delta s_k}=\lim_{\mu(P)\to 0} e^{\sum_{k=1}^n \operatorname{tr}A(s_k)\Delta s_k}$$
$$=e^{\int_a^x \operatorname{tr}A(s)\,ds}, \tag{1.27}$$

where we have used some elementary manipulations and the fact that the Riemann sum $\sum_{k=1}^n \operatorname{tr}A(s_k)\Delta s_k$ approaches $\int_a^x \operatorname{tr}A(s)\,ds$ as $\mu(P)\to 0$. Thus the theorem is proved. ∎

If y is a point of $[a,b]$, then by applying the above analysis to the interval $[y,b]$ instead of $[a,b]$, we can say that we have defined $\prod_y^x e^{A(s)\,ds}$ for all $x\in[y,b]$, that is to say for all $x\geqslant y$. It will be convenient also to define $\prod_y^x e^{A(s)\,ds}$ when $x<y$. We do this by analogy with the definition

$$\int_y^x f(s)\,ds=-\int_x^y f(s)\,ds \tag{1.28}$$

in the usual theory of integration. Equation (1.28) states that $\int_y^x f(s)\,ds$ is the additive inverse of $\int_x^y f(s)\,ds$. For product integrals, we merely replace "additive" by "multiplicative":

Definition 1.4. *Let $A:[a,b]\to\mathbb{C}_{n\times n}$ be continuous and let $x,y\in[a,b]$ with $x<y$. Then $\prod_y^x e^{A(s)\,ds}$ is defined by the formula*

$$\prod_y^x e^{A(s)\,ds}=\left(\prod_x^y e^{A(s)\,ds}\right)^{-1}. \tag{1.29}$$

Note that combining Definition 1.4 with Theorem 1.4 and Eq. (1.28), we obtain for any $x,y\in[a,b]$ the formula

$$\det\prod_y^x e^{A(s)\,ds}=e^{\int_y^x \operatorname{tr}A(s)\,ds}. \tag{1.30}$$

In the ordinary theory of integration one proves the following additive property of the integral:

$$\int_z^x f(s)\,ds=\int_y^x f(s)\,ds+\int_z^y f(s)\,ds. \tag{1.31}$$

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