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Excerpt
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Preliminary results

This chapter contains several elementary results from the theory of differential and integral equations, along with certain other preliminary results from classical analysis, and can be skimmed lightly during a first reading.

0.1 Variation of parameters

Consider the single (scalar) second-order linear differential equation

$$\mathscr{L}v = f(t) \quad \text{for real } t \text{ in the interval } [t_1, t_2], \tag{0.1.1}$$

where f is a given piecewise-continuous function on $[t_1, t_2]$, and the differential operator $\mathscr{L}$ is defined as

$$\mathscr{L}v(t) := \frac{d^2v}{dt^2} + a(t)\frac{dv}{dt} + b(t)v \tag{0.1.2}$$

for any twice-differentiable function v, where a and b are fixed, given piecewise-continuous functions on $[t_1, t_2]$. (When the notation $[t_1, t_2]$ is used to denote an interval, it will always be assumed that $t_1 \leq t_2$.)

If v_1 and v_2 are any two linearly independent solutions of the homogeneous equation $\mathscr{L}v = 0$, then *variation of parameters* can be used to represent the general solution of (0.1.1) in the form (for any fixed t_0 in $[t_1, t_2]$)

$$v(t) = c_1v_1(t) + c_2v_2(t) + \int_{t_0}^{t} K(t,s)f(s)\,ds, \tag{0.1.3}$$

where c_1 and c_2 are suitable constants, and the kernel K is defined as

$$K(t,s) := \frac{v_1(t)v_2(s) - v_2(t)v_1(s)}{v_1'(s)v_2(s) - v_2'(s)v_1(s)}. \tag{0.1.4}$$

The idea of variation of parameters was introduced by Isaac Newton, Gottfried Leibniz, and Johann Bernoulli independently in certain special situations, and the technique was later developed into a general procedure by Leonhard Euler and Joseph Lagrange. The procedure is much used in perturbation theory, as illustrated repeatedly in every chapter of this book (cf. Exercise 0.1.1).

Example 0.1.1: If $\mathscr{L}$ is the operator $\mathscr{L} = (d^2/dt^2) + 1$ (with $a = 0$, $b = 1$), then the functions v_1 and v_2 can be taken as $v_1(t) = \cos t$ and $v_2(t) = \sin t$, and the kernel K becomes $K(t,s) = \cos s \sin t - \sin s \cos t = \sin(t-s)$. The result (0.1.3) then leads to the identity (with $t_0 = 0$)

$$v(t) = c_1 \cos t + c_2 \sin t + \int_0^t \sin(t-s)[v''(s) + v(s)]\, ds$$

for suitable constants c_1 and c_2 depending on v.

Example 0.1.2: If $\mathscr{L}$ is the operator $\mathscr{L} = (d^2/dt^2) + a(t)(d/dt)$ (with $b = 0$), then the functions v_1 and v_2 can be taken as $v_1(t) = 1$ (for all t) and

$$v_2(t) = \int_{t_0}^t \exp\left(-\int_{t_0}^r a\right) dr,$$

and (0.1.3) leads to the result

$$v(t) = c_1 + c_2 K(t, t_0) + \int_{t_0}^t K(t,s)[v''(s) + a(s)v'(s)]\, ds,$$

with kernel

$$K(t,s) = \int_s^t \exp\left(-\int_s^r a\right) dr, \tag{0.1.5}$$

where the constants c_1 and c_2 depend on v.

We can use Example 0.1.2 to obtain an integral relation for any solution of the differential equation (0.1.1). Indeed, we can put $v''(s) + a(s)v'(s) = f(s) - b(s)v(s)$ in the result of Example 0.1.2 and find the equation

$$v(t) = c_1 + c_2 K(t,t_0) + \int_{t_0}^t K(t,s) f(s)\, ds - \int_{t_0}^t K(t,s) b(s) v(s)\, ds \tag{0.1.6}$$

for any solution of the differential equation

$$v'' + a(t)v' + b(t)v = f(t), \tag{0.1.7}$$

where K is given by (0.1.5). The constants c_1 and c_2 can be represented as

$$c_1 = v(t_0), \qquad c_2 = v'(t_0). \tag{0.1.8}$$

If $a = 0$, then the kernel K of (0.1.5) reduces to

$$K(t,s) = t - s \quad \text{if } a = 0. \tag{0.1.9}$$

Exercises

Exercise 0.1.1: Let $M_{m,n}(\mathbb{R})$ denote the collection of $m \times n$ matrices over the real numbers $\mathbb{R}$, let $A = A(t)$ be a continuous matrix-valued function on $[t_1, t_2]$ taking values in $M_{n,n}(\mathbb{R})$, let $f = f(t)$ be a continuous vector-valued function on $[t_1, t_2]$ taking values in $M_{n,1}(\mathbb{R})$, and consider the nonhomogeneous vector differential equation*

$$\frac{dx}{dt} = A(t)x + f(t) \quad \text{for } t \in [t_1, t_2] \tag{0.1.10}$$

for a solution vector $x = x(t)$ taking values in $M_{n,1}(\mathbb{R})$. Let $x^1 = x^1(t), x^2 = x^2(t), \dots, x^n = x^n(t)$ be any fundamental solution set (of solution vectors) for the homogeneous equation

$$\frac{dx}{dt} = A(t)x \quad \text{for } t \in [t_1, t_2] \tag{0.1.11}$$

(that is, the n solution vectors $x^1, x^2, \dots, x^n$ constitute a linearly independent set), and let t_0 be any fixed number in the given interval $[t_1, t_2]$. Derive the formula

$$x(t) = X(t)c + \int_{t_0}^{t} X(t)X(s)^{-1}f(s)\,ds \tag{0.1.12}$$

for the general solution of (0.1.10), where X is a fundamental matrix for (0.1.11) given as $X := [x^1, x^2, \dots, x^n]$, and c is a suitable constant vector depending on t_0, x, and X, with $c = X(t_0)^{-1}x(t_0)$. *Hint*: Seek x as a suitable linear combination of $x^1, x^2, \dots, x^n$ as

$$x = \sum_{j=1}^{n} x^j(t)c_j(t),$$

with scalar coefficients $c_j = c_j(t)$ depending on t, or equivalently $x = X(t)c(t)$ for some suitable vector-valued function $c = c(t)$.

Exercise 0.1.2: Let $A = A(t)$ and $f = f(t)$ be as in Exercise 0.1.1, and consider the *boundary-value problem* consisting of the differential equa-

* Note that vector-valued quantities are indicated in italics (i.e., x) rather than boldface (i.e., **x**) throughout this book.

tion (0.1.10) along with the following specified boundary condition on $x = x(t)$ at the endpoints t_1 and t_2:

$$Lx(t_1) + Rx(t_2) = \alpha, \tag{0.1.13}$$

where L and R are given (constant) $n \times n$ matrices and α is a given (constant) vector. Let $X(t)$ be a fundamental solution matrix for the homogeneous equation (0.1.11) as in Exercise 0.1.1. *Show that this boundary-value problem has a unique solution* (for every α and f) *if and only if the matrix*

$$M := LX(t_1) + RX(t_2) \tag{0.1.14}$$

is nonsingular, and in this case *show that the unique solution* $x(t)$ *can be given as*

$$x(t) = X(t)M^{-1}\alpha + \int_{t_1}^{t_2} G(t,s)f(s)\,ds, \tag{0.1.15}$$

where the *Green function* $G = G(t, s)$ is the matrix-valued function defined as

$$G(t,s) := \begin{cases} X(t)M^{-1}LX(t_1)X(s)^{-1} & \text{for } s < t, \\ -X(t)M^{-1}RX(t_2)X(s)^{-1} & \text{for } s > t. \end{cases} \tag{0.1.16}$$

If the matrix M is singular, *show that the boundary-value problem may have no solution or infinitely many solutions, depending on the data*. (This is the **Fredholm alternative.**) *Hint*: Put $t_0 = t_1$ in the general representation (0.1.12), and then insert this result for $x(t)$ into the boundary condition (0.1.13), so as to obtain for the constant vector c the result

$$Mc = \alpha - RX(t_2)\int_{t_1}^{t_2} X(s)^{-1}f(s)\,ds.$$

This last equation can be solved for c if the matrix M is nonsingular. The resulting value for c can be inserted back into (0.1.12), and one finds the stated representation (0.1.15). As a function of t, the Green function satisfies the homogeneous equation $\partial G(t,s)/\partial t = A(t)G(t,s)$ for $t \neq s$, along with the homogeneous boundary condition $LG(t_1, s) + RG(t_2, s) = 0$. The Green function has a jump discontinuity at $s = t$ characterized as $G(t, t-) - G(t, t+) = I =$ identity matrix. Finally, as a function of s, the Green function satisfies the adjoint equation $\partial G(t,s)/\partial s = -G(t,s)A(s)$. [See pp. 145–51 of R. Cole (1968) for further details.]

Exercise 0.1.3: The Bessel function J_0 satisfies the differential equation

$$v'' + \frac{1}{t}v' + v = 0 \tag{0.1.17}$$

subject to the initial conditions

$$v(0) = 1, \qquad v'(0) = 0. \tag{0.1.18}$$

Show that the solution v of (0.1.17)–(0.1.18) can be characterized equivalently as the solution of the integral equation

$$v(t) = 1 + \int_0^t s\left(\log\frac{s}{t}\right)v(s)\,ds. \tag{0.1.19}$$

Exercise 0.1.4: Derive the identity

$$\begin{aligned} v(t) &= v(t_0) + v'(t_0)(t - t_0) + v''(t_0)K(t, t_0) \\ &\quad + \int_{t_0}^t K(t,s)\left[\frac{d^3v(s)}{ds^3} + a(s)\frac{d^2v(s)}{ds^2}\right]ds \end{aligned} \tag{0.1.20}$$

for any function v of class C^3, where the Kernel $K = K(t, s)$ is defined as

$$K(t,s) := \int_s^t (t - r)\exp\left(-\int_s^r a\right)dr. \tag{0.1.21}$$

Hint: The functions

$$v_1(t) = 1, \qquad v_2(t) = t, \qquad v_2(t) = \int_{t_0}^t (t - r)\exp\left(-\int_{t_0}^r a\right)dr,$$

provide three linearly independent solutions of the homogeneous equation

$$\frac{d^3v}{dt^3} + a(t)\frac{d^2v}{dt^2} = 0.$$

Use these solutions along with variation of parameters to obtain the general solution of the inhomogeneous equation

$$\frac{d^3v}{dt^3} + a(t)\frac{d^2v}{dt^2} = f(t).$$

Exercise 0.1.5: Show that the initial-value problem

$$\begin{gathered} \frac{d^3v}{dt^3} + a(t)\frac{d^2v}{dt^2} + b(t)\frac{dv}{dt} + c(t)v = f(t) \quad \text{for } t \geq t_0, \\ v(t_0) = \alpha, \qquad v'(t_0) = \beta, \qquad v''(t_0) = \gamma, \end{gathered} \tag{0.1.22}$$

is equivalent to the following system of integral equations:

$$v(t) = \alpha + \beta\cdot(t - t_0) + \gamma\cdot K(t, t_0) + \int_{t_0}^{t} K(t,s)f(s)\,ds - \int_{t_0}^{t} K(t,s)[b(s)v'(s) + c(s)v(s)]\,ds \tag{0.1.23}$$

and

$$v'(t) = \beta + \gamma\cdot K_t(t, t_0) + \int_{t_0}^{t} K_t(t,s)f(s)\,ds - \int_{t_0}^{t} K_t(t,s)[b(s)v'(s) + c(s)v(s)]\,ds, \tag{0.1.24}$$

where $K = K(t,s)$ is given by (0.1.21), and $K_t(t,s) = \partial K(t,s)/\partial t$. *Hint*: The equation (0.1.23) follows from (0.1.20) and (0.1.22), and (0.1.24) follows from (0.1.23) on differentiation.

Exercise 0.1.6: Show that the solution v of the initial-value problem (0.1.22) can be given as

$$v(t) = \alpha + \beta\cdot(t - t_0) + \gamma\cdot K(t, t_0) + \int_{t_0}^{t} K(t,s)f(s)\,ds - \int_{t_0}^{t} K(t,s)u(s)\,ds, \tag{0.1.25}$$

where the function $u = u(t)$ is the solution of the equation

$$u(t) = P(t) + \int_{t_0}^{t} H(t,s)u(s)\,ds, \tag{0.1.26}$$

with kernel $H = H(t,s)$ defined as

$$H(t,s) := -[b(t)K_t(t,s) + c(t)K(t,s)], \tag{0.1.27}$$

and with the function $P = P(t)$ defined as

$$P(t) := b(t)p'(t) + c(t)p(t), \tag{0.1.28}$$

where $p = p(t)$ is given as

$$p(t) := \alpha + \beta\cdot(t - t_0) + \gamma\cdot K(t, t_0) + \int_{t_0}^{t} K(t,s)f(s)\,ds. \tag{0.1.29}$$

Hint: Define $u := bv' + cv$, and then take a suitable combination of equations (0.1.23) and (0.1.24) so as to obtain (0.1.26). Note that this linear Volterra integral equation (0.1.26) can be solved by Picard iteration, as described in Section 0.2. The approach of Exercises 0.1.4–0.1.6 can be extended in the obvious way to the study of the initial-value problem for the scalar nth-order linear differential equation.

0.2 The linear Volterra integral equation, and Gronwall's inequality

Consider the (scalar) integral equation

$$v(t) = p(t) + \int_{t_0}^{t} V(t,s)v(s)\,ds \tag{0.2.1}$$

for a real function v, where $p = p(t)$ and $V = V(t,s)$ are given piecewise-continuous functions for $t_0 \le t \le t_1$ and $t_0 \le s \le t \le t_1$, respectively.

The uniqueness of solutions to (0.2.1) follows immediately from the important **Gronwall's inequality**:

Theorem 0.2.1 (Gronwall 1919): *Let a, b, and c be nonnegative-valued continuous functions on the interval $[t_0, t_1)$, and let Z be a continuous function satisfying the integral inequality*

$$|Z(t)| \le a(t)\int_{t_0}^{t} b(s)|Z(s)|\,ds + c(t) \quad \text{on } [t_0, t_1). \tag{0.2.2}$$

Then Z also satisfies

$$|Z(t)| \le a(t)\int_{t_0}^{t} b(s)c(s)\exp\left(\int_s^t ab\right) ds + c(t) \tag{0.2.3}$$

on $[t_0, t_1)$.

Proof: Put

$$S(t) := \int_{t_0}^{t} b(s)|Z(s)|\,ds,$$

with $|Z(t)| \le a(t)S(t) + c(t)$, and find $S(t_0) = 0$ and $S'(t) = b(t)|Z(t)| \le a(t)b(t)S(t) + b(t)c(t)$. Hence, we find

$$\frac{d}{dt}\left\{\left[\exp\left(-\int_{t_0}^{t} ab\right)\right]S(t)\right\} \le \left[\exp\left(-\int_{t_0}^{t} ab\right)\right]b(t)c(t),$$

and integration leads to the inequality

$$\left[\exp\left(-\int_{t_0}^{t} ab\right)\right]S(t) \le \int_{t_0}^{t}\left[\exp\left(-\int_{t_0}^{s} ab\right)\right]b(s)c(s)\,ds,$$

from which the stated result follows easily. ■

A key step in this proof of Gronwall's inequality involves the integration of a certain *differential inequality*. We shall see in Section 0.3 and in later chapters that related integrations of various differential inequalities play important roles in the study of a wide range of linear and nonlinear singularly perturbed problems.

Gronwall's inequality implies that *(0.2.1) has at most one solution*. Indeed, if v_1 and v_2 are solutions, then $Z := v_1 - v_2$ is seen to satisfy (0.2.2), with $b = 1$, $c = 0$, and $a = M$ for some suitable positive *constant* M. Then (0.2.3) implies the result $|v_1 - v_2| \leq 0$, or $v_1 = v_2$, so that any two solutions coincide. [Note that any solution of (0.2.1) must be piecewise-continuous, and the required version of Gronwall's inequality for piecewise-continuous Z is easily seen to be valid.]

The unique solution of the Volterra equation (0.2.1) can be obtained by **Picard's method of successive approximations** as (see Exercises 0.2.1–0.2.2)

$$v(t) = p(t) + \int_{t_0}^{t} V^*(t,s)p(s)\,ds, \tag{0.2.4}$$

where V^* is the Volterra *resolvent kernel* for V, defined as

$$V^*(t,s) := \sum_{j=1}^{\infty} V_j(t,s), \tag{0.2.5}$$

with

$$V_j(t,s) := \begin{cases} V(t,s) & \text{if } j = 1, \\ \int_s^t V(t,r)V_{j-1}(r,s)\,dr & \text{for } j = 2,3,\ldots. \end{cases} \tag{0.2.6}$$

The series (0.2.5), which defines the resolvent kernel, can be shown to be absolutely and uniformly convergent for $t_0 \leq s \leq t \leq t_1$, and there holds

$$|V^*(t,s)| \leq M\exp(M|t-s|), \tag{0.2.7}$$

where the constant M can be taken to be any upper bound on $|V|$.

The method of successive approximations seems to have been introduced into the study of differential and integral equations by Liouville (1837). Many workers, including E. Picard, C. Neumann, V. Volterra, and S. Banach, have forged the technique into a powerful, general method culminating in the important *Banach/Picard fixed-point theorem* discussed in Section 0.4. [See pp. 719–20, 1054–7, and 1089–90 of Kline (1972).]

The unique solvability of the integral equation (0.2.1) implies also the unique solvability of the Cauchy problem

$$v'' + a(t)v' + b(t)v = f(t) \quad \text{for } t > t_0,$$
$$v(t_0) = c_1, \qquad v'(t_0) = c_2, \tag{0.2.8}$$

for given constants c_1 and c_2 and for given piecewise-continuous functions a, b, and f. Indeed, the problem (0.2.8) is equivalent to the integral equation (0.1.6), and this latter integral equation is of the form (0.2.1), with

$$\begin{aligned} p(t) &:= c_1 + c_2 K(t, t_0) + \int_{t_0}^{t} K(t,s) f(s)\, ds, \\ V(t,s) &:= -K(t,s) b(s), \end{aligned} \tag{0.2.9}$$

with K given by (0.1.5). Hence, there is one and only one solution v to (0.2.8), and this solution can be given by the appropriate representation (0.2.4).

The situation is somewhat more subtle regarding the existence and uniqueness of solutions for *boundary-value problems*. For example, the differential equation

$$v'' + v = 1 \tag{0.2.10}$$

always has a unique solution on the interval $[0, \pi/2]$ subject to the following general Dirichlet boundary conditions:

$$v(0) = v_0, \qquad v(\pi/2) = v_1, \tag{0.2.11}$$

for any given constants v_0 and v_1. However, the same differential equation (0.2.10) has *no solution* on the interval $[0, \pi]$ subject to the boundary conditions

$$v(0) = 0, \qquad v(\pi) = 1, \tag{0.2.12}$$

whereas (0.2.10) has *infinitely many solutions* on $[0, \pi]$ subject to the conditions

$$v(0) = 0, \qquad v(\pi) = 2. \tag{0.2.13}$$

By way of comparison, the Dirichlet problem for the equation

$$v'' - v = f(t) \tag{0.2.14}$$

always has one and only one solution on any given interval.

Cochran (1968) has shown that the earlier results of this section on the initial-value problem can also be used in the study of boundary-value problems. An example of Cochran's results is given in Section 8.1.

Exercises

Exercise 0.2.1: Prove that the infinite series (0.2.5) converges absolutely and uniformly for $t_0 \le s \le t \le t_1$, and derive the inequality (0.2.7).

Exercise 0.2.2: Prove that the function defined by the right side of (0.2.4) provides a solution to the Volterra integral equation (0.2.1). (Gronwall's inequality implies that the solution is unique.)

Exercise 0.2.3: Use the definition (0.2.5)–(0.2.6) to compute the resolvent kernel V^* for $V(t,s) := t - s$. Use this result along with (0.2.4) to solve the equation

$$v(t) = 1 + t + \int_0^t (t-s)v(s)\,ds.$$

[This integral equation is equivalent to the initial-value problem $v'' - v = 0$, $v(0) = v'(0) = 1$; hence, v can be obtained more simply by any one of several other methods, such as Laplace transformation or variation of parameters.]

Exercise 0.2.4: Consider the function V defined as $V(t,s) := a(t)b(s)$ for two given piecewise-continuous functions a and b. Show that the Volterra resolvent for this kernel can be given as

$$V^*(t,s) = a(t)b(s)\exp\left(\int_s^t ab\right).$$

[This shows that Gronwall's inequality is sharp, in the sense that equality in (0.2.2) implies also equality in (0.2.3).]

Exercise 0.2.5: The Bessel function $J_0(t)$ can be characterized as the solution of the Volterra equation [see (0.1.19)]

$$v(t) = 1 + \int_0^t s(\log s - \log t)v(s)\,ds. \tag{0.2.15}$$

Solve (0.2.15) directly by iteration or successive approximations to find

$$v(t) = J_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k}}{(k!)^2}.$$

Hint: Seek v as the limit

$$v(t) = \lim_{k\to\infty} v_k(t),$$

with $v_1(t) := 1$ and

$$v_{k+1}(t) := 1 + \int_0^t s(\log s - \log t)v_k(s)\,ds$$

for $k = 1, 2, \ldots$. The indefinite integral

$$\int x^k(\log x)\,dx = x^{k+1}\left[-(k+1)^{-2} + (k+1)^{-1}(\log x)\right]$$

will be helpful.

Exercise 0.2.6: Verify directly the validity of the assertions given in the text regarding (0.2.10)–(0.2.13).