1. How to make surfaces and talk about them

1.1 WHAT IS A SURFACE?

What do *you* mean by a Ⓣ surface? In real life we are surrounded by surfaces – those of furniture, tools, utensils, buildings, fluids, our bodies – and yet most people have surprising difficulty in being able to say what they mean by a surface in general. They know a particular surface when they see one, but how can we tell a computer or a blind man what it is about surfaces that they all have in common? Mathematicians began to be faced with this problem in the late nineteenth century as mathematics grew more complicated, and it took them many years to find a way of saying what they meant by a surface, that all mathematicians would understand. They had to be able to agree on what they were talking about before they could begin to *prove* things about surfaces, to do mathematics about surfaces. Their agreed statement about what a surface is, is called a *definition*; but it is not easy for beginners, so we shall approach the question in a different way. We shall eventually *make* a definition of our own, but in ordinary language that does not look mathematical. One can do mathematics in many dialects of a ‘professional’ language, and a mathematician chooses a particular dialect to suit his immediate purposes. Indeed, the following discussion is designed to show how mathematicians look at things they wish to study, and decide on the right words to use, in order to make their study easier.

Let us therefore ask a different question. How would you *make* a surface? (This might then help you to say what you mean by a surface: anything assembled according to your instructions would be a surface, although perhaps some exceptional ‘surfaces’ would not be made that way.) Now, most people only make surfaces as the ‘skin’ on some solid, by baking dough, moulding clay, or assembling wood or concrete forms. It is hard to say what we mean by a ‘solid’ and its ‘skin’ (has a jelly a surface; is it a solid?). But a seamstress makes a sort of surface when she stitches together pieces of cloth to make a dress, and an engineer makes a surface when he joins metal sheets to make the hull of a ship, the fuselage of a plane, or the body of a car. In all these cases, certain simple bits of surface – pieces of cloth, panels of metal – are being joined to make more complicated ones. We may not have the skill or tools of a seamstress or an engineer, but instead of pieces of cloth or metal we can use sheets of paper cut into polygons. These polygonal panels can be joined with sticky tape along edges, instead of being stitched or welded, to form more complicated things that most people will agree are surfaces. Some people might say that these paper surfaces are rather special, for various reasons, but let us consider

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the objections later, when we have considered such paper surfaces more closely.

1.2 TWO RULES FOR MAKING PAPER SURFACES

The simplest such paper surface, then, will be a single paper sheet, shaped into a polygon with at least three sides, but perhaps four, five, or more. Let us indicate the edges clearly by sticking tape (coloured black, say) along each one, as in Fig. 1.1; if we double the tape over the edge, the black stripe appears half-width. A polygon prepared in this way will be called a 'panel'.

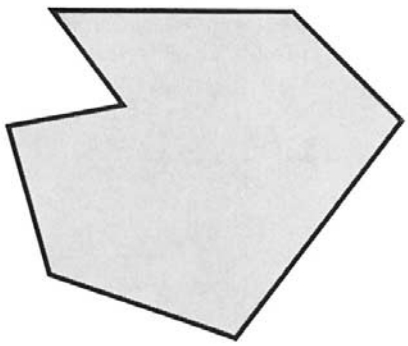


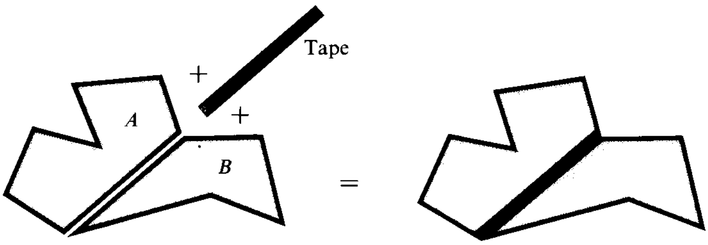
Fig. 1.1

Next, we can use the black tape to join two such panels along an edge, as in Fig. 1.2, and gradually build up constructions like those in Fig. 1.3. On each single panel, the tape appears half-width. Notice that in Fig. 1.2(*b*), we allow several edges of a new panel to be taped to edges of panels already assembled.

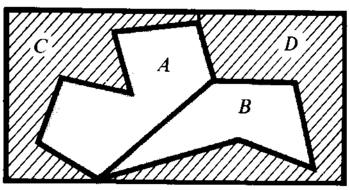
After some trials, we find we can bend our panels to make curved surfaces like that of the cylinder, or have three (or more!) panels with a common edge; see Fig. 1.3(*b*). Also we could use the black tape at a corner to make the constructions of Fig. 1.3(*c*), but this uses a different rule of construction from that shown in Fig. 1.2. To be quite clear about what we are allowing, let us lay down a rule of construction just as we would lay down rules if we were playing a game (we shall add another rule in a moment):

Rule 1
We can tape panels together only along their edges.

This rule, then, does not allow us to stick a panel by a corner, or to the middle of another. We could not, therefore, make the models shown in Fig. 1.3(*c*); and we cannot have Ⓣanything like *A*, *B*, *C* in Fig. 1.4(*a*). Instead we must get the same result as in Fig. 1.4(*a*), by following the order of Fig. 1.4(*b*). It will simplify things later if we take Rule 1 to mean that we *cannot* tape together two edges of the same panel.

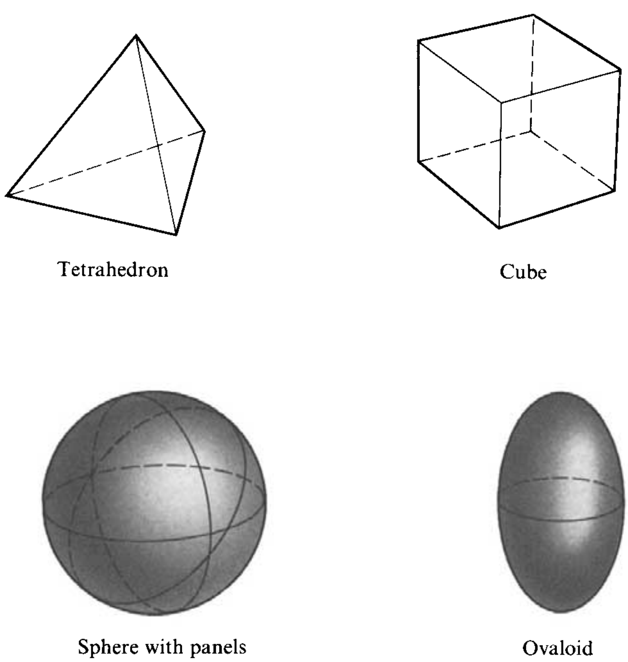


(a)



C has several edges joined to $A + B$
D has several edges joined to $A + B + C$

(b)



(c)

Fig. 1.2

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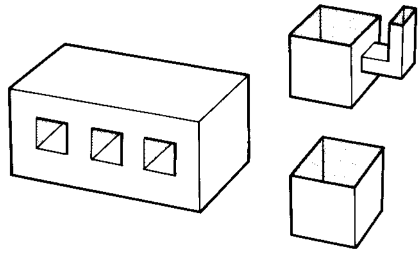
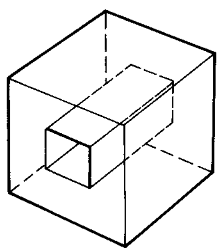
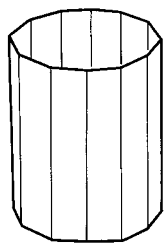
(*a*)



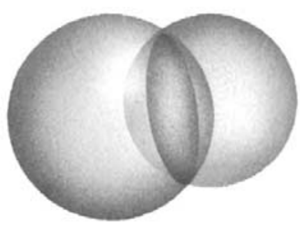


Block with tunnel (torus)



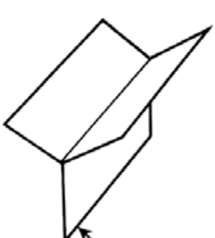
Curved surface

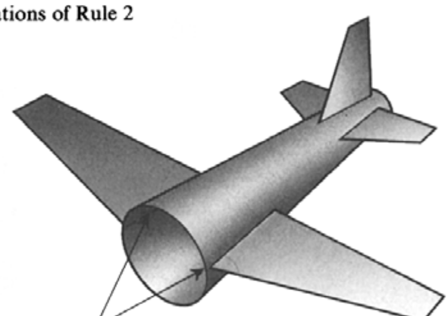
(*b*)



Soap bubbles intersecting

Violations of Rule 2





3 panels with common edge

Violations of Rule 2

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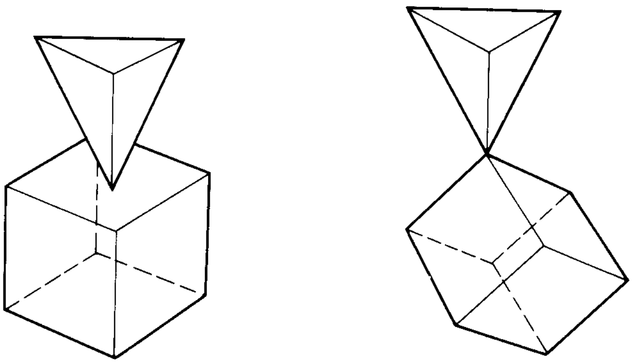
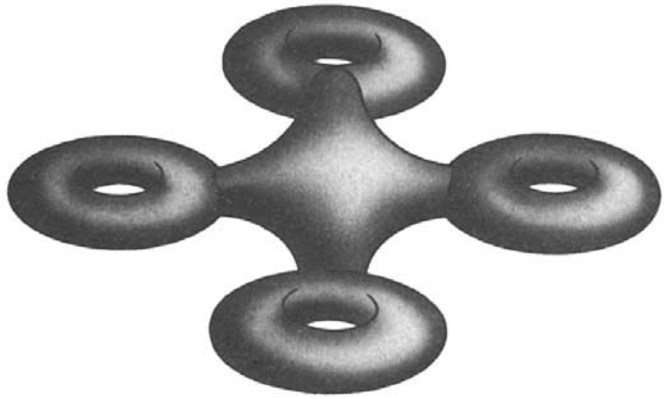


(*c*)

Violations of Rule 1

Torus

Double torus
(pretzel) with hole
(or puncture)



(*d*)

Surface of genus 4

Fig. 1.3

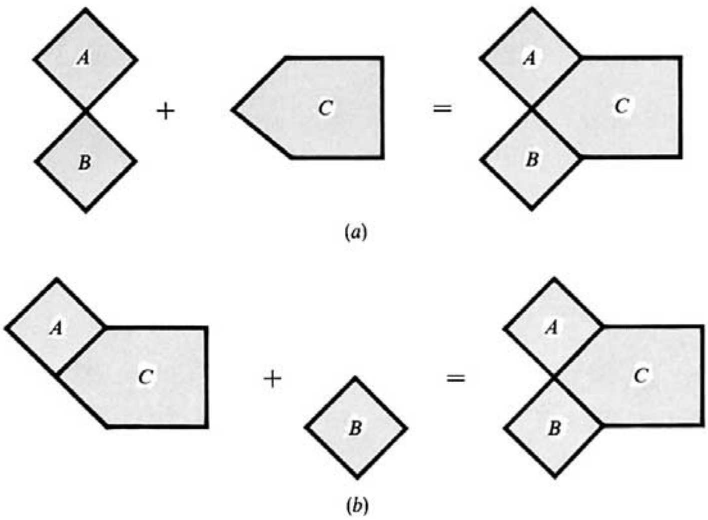


Fig. 1.4. (*a*) Wrong order; panels *A* and *B* were not taped along an edge. (*b*) Right order; an edge taped each time.

Many people will agree that these assemblies are good examples of surfaces, but some might make objections like this:

First Objection. 'These paper surfaces are rough, and the black tape sticks up against your hand when you stroke the surface.'

To which we could reply that by being sufficiently careful, and using sufficiently flimsy tape, we could make the panels as smooth as we like.

Second Objection. 'These paper surfaces have sharp corners and angles, but most surfaces we see are smooth like spheres or glass jars.'

To which we could reply that if we used enough *very tiny* panels, we could smooth out these surfaces as much as we liked, so that they would feel quite smooth. (In fact, many 'rounded' surfaces around us are made of millions of tiny crystals, as in the frontispiece, which have flat triangular or square faces.) So, *by using enough panels*, we can model most surfaces rather well by these constructions – in particular, those of Fig. 1.3(*d*).

Indeed, we could now model such a great variety of surfaces with these panels, that we shall need to exclude certain kinds, for simplicity. The ones where three or more panels are joined by an edge could be quite complicated, like the soap bubble in Fig. 1.3; but that kind of complication is not what we want to talk about at the moment. We therefore leave them out by agreeing to impose a second rule:

Rule 2
Any edge of the paper surface can belong to either one or two panels, but not to three or more.

Thus, once two panels have been joined by tape along an edge, that edge cannot be joined to any further panel. This rule excludes surfaces like that of Fig. 1.3(*b*), but our two Rules, 1 and 2, still allow us to make a very wide range of surfaces. We now want to describe these, and find out things about them.

Let us call Ⓣ[a] any assembly of panels, taped together according to both Rules 1 and 2, a '**paper surface**'. It will consist of a finite number of panels, with *edges* and *corners* indicated by the black tape. (These edges and vertices form what is known as a 'linear graph', Ⓣ[b] whose nodes are the corners of the panels. We shall not need the theory of linear graphs in this book, but the interested reader may consult Ore [15].)

1.3 SOME NAMES FOR THINGS

When we make paper surfaces according to our two rules, it helps to have names for some special parts, and for some of our more common surfaces. We have already mentioned the *panels*, the *edges* and the *corners*. By Rule 2, an edge belongs to just one panel, or to just two. The first kind of edge is called **free**, and we have no special name for the other kind – we can call them 'not free' if we need to talk about them. If there are no free edges at all, we call the surface **closed**. If there are some free edges, however, they form the **boundary** of the surface. Let us look at some examples.

The single panel in Fig. 1.1 has all its edges free, and its boundary is indicated by the heavy lines. It is customary to give the names 'tetrahedron', 'cube', 'torus', and 'surface of genus 4' respectively to the surfaces thus labelled in Figs. 1.2 and 1.3; each is a closed surface. We shall later say more about the strange name 'genus', but it refers to the numbers of 'handles' of which there are 4 in the sketch. The second surface in Fig 1.3(*d*) has a boundary and is called a 'double torus with one hole'.

Exercise 1.3

1. What do you think we would mean by a 'triple torus with five holes'? Sketch a quintuple torus with no holes. Is it a surface of genus 5, would you say?

2. For each surface S of those illustrated in Figs. 1.2 and 1.3(*a*) count the numbers of corners, edges and panels in S. If these numbers are C, E and P respectively, find the number $C - E + P$ in each case. (This curious alternating sum is interesting, as we shall see later.) Mark in panels on Fig. 1.3(*d*) and calculate similarly.

It might happen that the panels of a surface all have the same number of edges, while equal numbers of panels meet at each corner. The surface is then called a 'regular polyhedron'. For example, the tetrahedron and cube are regular polyhedra. More examples of regular polyhedra are shown in Fig. 1.5, and they look especially beautiful if all their panels are exactly the same shape,* with all edges and angles equal. (Their boundaries are then called 'regular polygons'.) You can understand the names better if you know that the ancient Greeks studied the regular polyhedra and called them the 'Platonic solids' after the famous philosopher Plato (fourth century B.C.) who thought about them a lot. (We shall avoid the word 'solid' here.) The Greeks called the panels 'hedra', so 'tetrahedron' means

* If these are merely quadrilaterals, the surface is called a hexahedron or cuboid, rather than a cube.

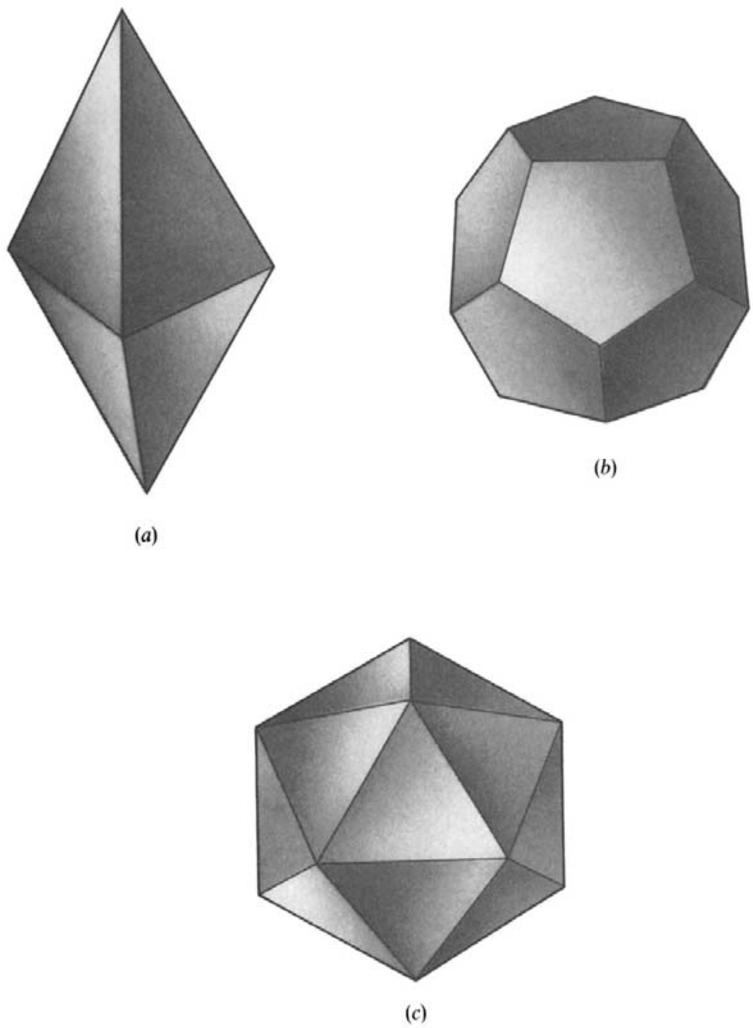


Fig. 1.5. The Platonic polyhedra other than cube and tetrahedron: (*a*) octahedron, (*b*) dodecahedron, (*c*) icosahedron.

'4-face', and the plural is 'tetrahedra'. The other names should now make sense, if we remember that 'octa' means 'eight', 'hexa' means 'six', 'icosa' means 'twenty', and 'dodeca' means 'twelve' (= 'two-ten'). These Platonic polyhedra are all *closed*, because they have no free edges. An interesting question asked by the Greeks was this:

'Are there any other kinds of regular polyhedron?'

and they were able to answer 'No', at least with a certain other restriction. We shall look at their explanation later (see Exercise 2.6, No. 13).

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1.4 SOME SIMPLE SURFACES WITH BOUNDARY

A very simple surface, which is not closed, is the **cylinder** in Fig. 1.3(*b*). It has some very close relatives, for example, the **prisms** of Fig. 1.6; and another

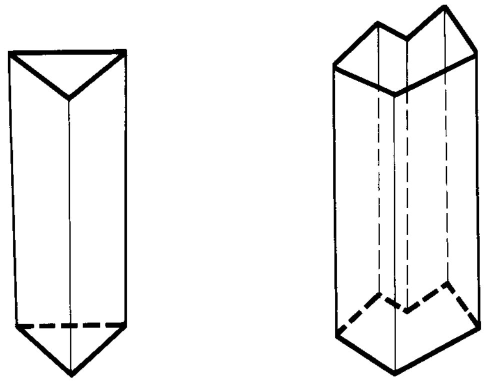

Fig. 1.6. Prisms.

surface in the same family is the **annulus**, which is like a disc with a hole punched in it. ('Annulus' is a Latin word meaning 'ring', and most people think of an annulus as circular, as in Fig. 1.7(*a*); but we use the name for any shape that looks like a panel with a single hole, even if it is made with

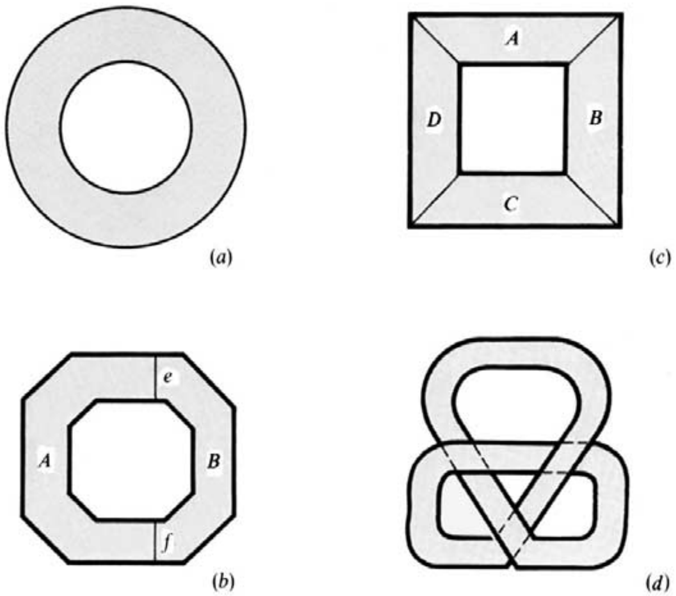

Fig. 1.7. Annuli.

several panels like those in Fig. 1.7(*b*) and (*c*).) The cylinder, the prisms and the annuli all have free edges, and so *they are not closed surfaces*. Each has a boundary, consisting of two endless chains of free edges. Each such chain forms a loop which does not cross itself. We call such loops **Jordan curves** after the French mathematician C. Jordan (1838–1922) who first studied them. If a surface has one or more boundary curves, we call it a **surface with boundary**; thus a surface is either closed, or else it is a surface with boundary.

Let us look again at the annulus in Fig. 1.7(*b*). It is very simple, because it has only the two panels *A* and *B*, taped together along the edges *e* and *f*. Now, if we asked a friend to do the taping, he could give us a surprise we didn't expect! For, after taping along the edge *e*, he might give the panel *B* a twist before he taped the edge *f*. The result would look like one of the pair shown in Fig. 1.8,* and it is called a **Moebius band** (or Moebius **strip**) after the German mathematician A. F. Moebius (1790–1868) who first wrote about it. Notice that it is not flat like the annulus and has only *one* Jordan curve for its boundary. The two types of band in Fig. 1.8 arise because our

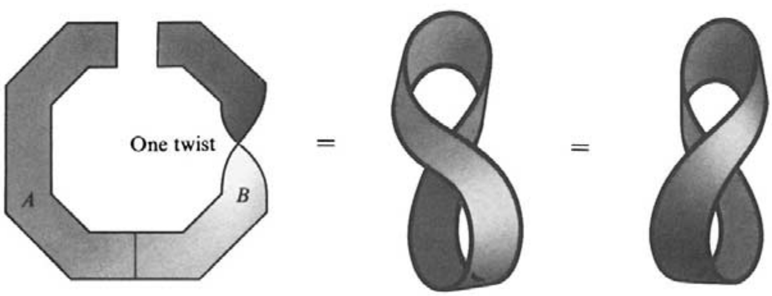

Fig. 1.8. Moebius band.

friend could have given the panel *B* a left-hand twist, or a right-hand twist. If we reflect either of the two bands in a mirror, the image will be the other band, so they resemble each other in the way our left hand resembles our right. Because of this, we shall not mind which direction of twist is given.

You can check that this kind of effect would occur also if the panel *B* had been twisted not 1 but 3, 5, or any *odd* number of times. If *B* has been twisted 2, 4, or any *even* number of times, the result would be a surface with two boundary curves, just as with the annulus in Fig. 1.7(*b*), where *B* was twisted no times (still an even number of times!). Usually we shall only think of a panel with no twist or one twist, and as we said above, we ignore Ⓣ the direction of the twist.

1.5 THINKING ABOUT A MOEBIUS BAND

The Moebius band has some strange properties when we compare it with the annulus, and these properties are sometimes shown off by conjurors.

* The reader is not expected to draw for himself pictures of this quality each time we study a Moebius band. We shall later use an easier and more schematic type of drawing, but each time an unfamiliar surface is introduced, we use a realistic picture for clarity.