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1

THE LANGUAGE OF FUNCTORS

1.1 Notation

Λ, Γ, Δ will denote rings with identity elements. They need not be commutative. Z will denote the ring of integers. The category of left (resp. right) Λ-modules will be denoted by $\mathscr{C}_\Lambda^L$ (resp. $\mathscr{C}_\Lambda^R$). Sometimes it is immaterial whether we work exclusively with left Λ-modules or exclusively with right Λ-modules. In such a case $\mathscr{C}_\Lambda$ will denote the category in question. When Λ is commutative, we make no distinction between $\mathscr{C}_\Lambda^L$ and $\mathscr{C}_\Lambda^R$. Also we normally identify the category of additively written abelian groups with the category of Z-modules. Finally i_A is used to denote the identity map of A.

1.2 Bimodules

Suppose that A is both a Λ-module and a Γ-module, the additive structure being the same in both cases. Let us suppose that multiplication (of an element of A) by an element of Λ always commutes with multiplication by an element of Γ. We then say that A is a (Λ, Γ)-*bimodule*. If, for example, Λ operates on the left and Γ on the right, we may indicate this by writing ${}_\Lambda A_\Gamma$. If A and A' are both (Λ, Γ)-bimodules of the same type, then a mapping $f: A \to A'$ which is simultaneously Λ-linear and Γ-linear is called a *bihomomorphism*.

***Example* 1.** Every Λ-module is a (Λ, Z)-bimodule.

***Example* 2.** If Γ is the centre of Λ, then every Λ-module is a (Λ, Γ)-bimodule.

***Example* 3.** Λ itself is a (Λ, Λ)-bimodule with one Λ acting on the right and the other on the left. This is by virtue of the associative law of multiplication.

1.3 Covariant functors

Suppose that with each module A in $\mathscr{C}_\Lambda$ there is associated a module $F(A)$ in $\mathscr{C}_\Delta$ and that to each Λ-homomorphism $f: A \to A'$ there cor-

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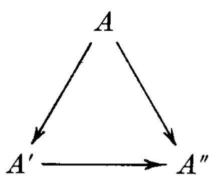
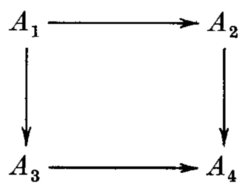
responds a Δ-homomorphism $F(f): F(A) \to F(A')$. Suppose further that

(1) $F(i_A) = i_{F(A)}$ *for all* A *in* $\mathscr{C}_\Lambda$;
(2) $F(gf) = F(g)\,F(f)$ *whenever* $f: A \to A'$ *and* $g: A' \to A''$ *in* $\mathscr{C}_\Lambda$.

In these circumstances we say we have a *covariant functor* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ from Λ-modules to Δ-modules. Simple commutative diagrams (of Λ-modules and Λ-homomorphisms) such as



and



remain commutative when a covariant functor is applied. Also if $f: A \to A'$ is an isomorphism and $g: A' \to A$ is its inverse, then, for a covariant functor F, $F(f): F(A) \to F(A')$ is an isomorphism and $F(g): F(A') \to F(A)$ is its inverse. This is because gf and fg are identity maps.

For the remainder of section (1.3), $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ will denote a covariant functor.

Definition. *F is said to be 'additive' if whenever $f_1: A \to A'$ and $f_2: A \to A'$ are Λ-homomorphisms, sharing a common domain A and a common codomain A', we have $F(f_1+f_2) = F(f_1)+F(f_2)$.*

Note. The Λ-homomorphisms of A into A' form an abelian group. This is denoted by $\mathrm{Hom}_\Lambda(A, A')$. Addition in $\mathrm{Hom}_\Lambda(A, A')$ is defined by $(f_1+f_2)(a) = f_1(a)+f_2(a)$.

If F is additive, then it carries null homomorphisms and null modules into null homomorphisms and null modules.

In the classical theory of modules, *finite* direct sums and *finite* direct products are indistinguishable. Here this is recognized by introducing the notion of a *biproduct*.

Let $A_1, A_2, \ldots, A_n$ and A be Λ-modules and suppose we are given homomorphisms $\sigma_i: A_i \to A\ (1 \leqslant i \leqslant n)$ and $\pi_i: A \to A_i\ (1 \leqslant i \leqslant n)$. The complete system is called a representation of A as a *biproduct* of $A_1, A_2, \ldots, A_n$ if

(*a*) $\pi_j\sigma_i = \delta_{ji}$, i.e. $\pi_j\sigma_i$ is a null resp. identity homomorphism if $i \neq j$ resp. $i = j$;
(*b*) $\Sigma\sigma_i\pi_i =$ identity.

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In these circumstances we write variously

$$A = A_1 \oplus A_2 \oplus \dots \oplus A_n \text{ (direct sum notation)},$$
$$A = A_1 \times A_2 \times \dots \times A_n \text{ (direct product notation)},$$
$$A = A_1 * A_2 * \dots * A_n \text{ (biproduct notation)},$$

and, more explicitly,

$$[\sigma_1, \dots, \sigma_n; A; \pi_1, \dots, \pi_n] = A_1 * A_2 * \dots * A_n.$$

We call $\sigma_i: A_i \to A$ the *canonical injection* (it is necessarily a monomorphism) and $\pi_i: A \to A_i$ the *canonical projection* (it is necessarily an epimorphism).

Exercise 1.† *Let* $[\sigma_1, \dots, \sigma_n; A; \pi_1, \dots, \pi_n] = A_1 * A_2 * \dots * A_n$ *in* $\mathscr{C}_\Lambda$. *Show that if* Λ*-homomorphisms* $f_i: A_i \to B$ $(1 \leqslant i \leqslant n)$ *are given, then there exists a unique homomorphism* $f: A \to B$ *such that* $f\sigma_i = f_i$ *for* $1 \leqslant i \leqslant n$. *Show also that if* $g_i: B \to A_i$ $(1 \leqslant i \leqslant n)$ *are prescribed* Λ*-homomorphisms, then there exists a unique homomorphism* $g: B \to A$ *such that* $\pi_i g = g_i$ *for* $1 \leqslant i \leqslant n$.

Exercise 2. *Let* $[\sigma_1, \dots, \sigma_n; A; \pi_1, \dots, \pi_n] = A_1 * A_2 * \dots * A_n$ *in* $\mathscr{C}_\Lambda$. *Show that the homomorphism* $A_1 \oplus A_2 \oplus \dots \oplus A_n \to A$ *induced by the* σ_i *and the homomorphism* $A \to A_1 \times A_2 \times \dots \times A_n$ *induced by the* π_i *are both of them isomorphisms.*

Observe that if $A_1, A_2, \dots, A_n$ are given, then we can always find $A, \sigma_1, \sigma_2, \dots, \sigma_n$ and $\pi_1, \pi_2, \dots, \pi_n$ so that

$$[\sigma_1, \dots, \sigma_n; A; \pi_1, \dots, \pi_n] = A_1 * A_2 * \dots * A_n.$$

Theorem 1. Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be an additive covariant functor and let* $[\sigma_1, \dots, \sigma_n; A; \pi_1, \dots, \pi_n] = A_1 * A_2 * \dots * A_n$ *in* $\mathscr{C}_\Lambda$. *Then*
$$[F(\sigma_1), \dots, F(\sigma_n); F(A); F(\pi_1), \dots, F(\pi_n)] = F(A_1) * F(A_2) * \dots * F(A_n)$$
in $\mathscr{C}_\Delta$.

Proof. Apply F to the relations $\pi_j \sigma_i = \delta_{ji}$ and $\Sigma\, \sigma_i \pi_i =$ identity.

We shall now show that this property characterizes additive covariant functors.

Theorem 2. *Let* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be a covariant functor and suppose that whenever* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$ *in* $\mathscr{C}_\Lambda$, *then*

$$[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)] = F(A_1) * F(A_2) \text{ in } \mathscr{C}_\Delta.$$

In these circumstances F *is additive.*

† Solutions to the Exercises will be found at the end of the chapter.

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Proof. Let $f_1, f_2: A \to B$ be homomorphisms. Further, let

$$[\sigma_1, \sigma_2; C; \pi_1, \pi_2] = A * A.$$

Then $[F(\sigma_1), F(\sigma_2); F(C); F(\pi_1), F(\pi_2)] = F(A) * F(A)$ and therefore $i_{F(C)} = F(\sigma_1) F(\pi_1) + F(\sigma_2) F(\pi_2)$. Define $d: A \to C$ by $d = \sigma_1 + \sigma_2$. Then $\pi_1 d = \pi_1(\sigma_1 + \sigma_2) = i_A$ from which we obtain

$$F(\pi_1) F(d) = F(\pi_1 d) = F(i_A) = i_{F(A)}.$$

Similarly $F(\pi_2) F(d) = i_{F(A)}$. Now

$$F(d) = i_{F(C)} F(d) = (F(\sigma_1) F(\pi_1) + F(\sigma_2) F(\pi_2)) F(d).$$

Hence

$$\begin{aligned} F(d) &= F(\sigma_1) F(\pi_1) F(d) + F(\sigma_2) F(\pi_2) F(d) \\ &= F(\sigma_1) i_{F(A)} + F(\sigma_2) i_{F(A)} = F(\sigma_1) + F(\sigma_2). \end{aligned}$$

Define $g: C \to B$ by $g = f_1 \pi_1 + f_2 \pi_2$. Then

$$g\sigma_1 = (f_1 \pi_1 + f_2 \pi_2) \sigma_1 = f_1 \pi_1 \sigma_1 + f_2 \pi_2 \sigma_1 = f_1.$$

Similarly $g\sigma_2 = f_2$. Furthermore $gd = (f_1\pi_1 + f_2\pi_2)(\sigma_1 + \sigma_2) = f_1 + f_2$. Accordingly $F(f_1 + f_2) = F(gd) = F(g) F(d) = F(g)(F(\sigma_1) + F(\sigma_2))$. Thus

$$\begin{aligned} F(f_1 + f_2) &= F(g) F(\sigma_1) + F(g) F(\sigma_2) \\ &= F(g\sigma_1) + F(g\sigma_2) \\ &= F(f_1) + F(f_2). \end{aligned}$$

Hence f is additive.

Theorem 3. *Suppose that* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2) = A_1 * A_2$ *in* $\mathscr{C}_\Lambda$. *Then the sequences*

$$0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0 \qquad (1.3.1)$$

and

$$0 \to A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1 \to 0 \qquad (1.3.2)$$

are exact.

Proof. We need only consider (1.3.1) and for this it suffices to show $\operatorname{Ker} \pi_2 \subseteq \operatorname{Im} \sigma_1$. Let $\alpha \in \operatorname{Ker} \pi_2$. Then

$$\alpha = \sigma_1 \pi_1(\alpha) + \sigma_2 \pi_2(\alpha) = \sigma_1 \pi_1(\alpha) \in \operatorname{Im} \sigma_1.$$

Lemma 1. *Suppose that* $A_1 \xrightarrow{\sigma_1} A$ *and* $A \xrightarrow{\pi_1} A_1$ *are* Λ*-homomorphisms such that* $\pi_1 \sigma_1 =$ *identity. Then* $A = \operatorname{Im} \sigma_1 \oplus \operatorname{Ker} \pi_1$.

Proof. Let $a \in A$. Then $\pi_1(a - \sigma_1 \pi_1(a)) = 0$ and therefore

$$a = \sigma_1 \pi_1(a) + (a - \sigma_1 \pi_1(a)) \in \operatorname{Im} \sigma_1 + \operatorname{Ker} \pi_1.$$

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Now assume that $\alpha \in \operatorname{Im} \sigma_1 \cap \operatorname{Ker} \pi_1$, say $\alpha = \sigma_1(a_1)$ with $a_1 \in A_1$. Then

$$a_1 = \pi_1 \sigma_1(a_1) = \pi_1(\alpha) = 0$$

whence $\alpha = 0$. This shows that $A = \operatorname{Im} \sigma_1 \oplus \operatorname{Ker} \pi_1$.

Theorem 4. *Let* $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ *be an exact sequence in* $\mathscr{C}_\Lambda$. *Then the following statements are equivalent:*

(1) $\operatorname{Im} \sigma_1 (= \operatorname{Ker} \pi_2)$ *is a direct summand of* A;
(2) *there exists a* Λ*-homomorphism* $\pi_1 : A \to A_1$ *such that* $\pi_1 \sigma_1 =$ *identity*;
(3) *there exists a* Λ*-homomorphism* $\sigma_2 : A_2 \to A$ *such that* $\pi_2 \sigma_2 =$ *identity*;
(4) *there exist* Λ*-homomorphisms* $\sigma_2 : A_2 \to A$ *and* $\pi_1 : A \to A_1$ *such that* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$.

Proof. By the definitions and Lemma 1,

$$(4) \Rightarrow (2) \Rightarrow (1) \quad \text{and} \quad (4) \Rightarrow (3) \Rightarrow (1).$$

Assume (1), say $A = \operatorname{Im} \sigma_1 \oplus B$ for some submodule B of A. Now σ_1 induces an isomorphism $A_1 \xrightarrow{\sim} \operatorname{Im} \sigma_1$. Let $u : \operatorname{Im} \sigma_1 \xrightarrow{\sim} A_1$ be its inverse. Next π_2 induces an isomorphism $B \xrightarrow{\sim} A_2$. Let $v : A_2 \xrightarrow{\sim} B$ be its inverse. Put $\pi_1 = up$ and $\sigma_2 = jv$, where $p : A \to \operatorname{Im} \sigma_1$ is the projection associated with the relation $A = \operatorname{Im} \sigma_1 \oplus B$ and $j : B \to A$ is an inclusion mapping. Then $\pi_1 \sigma_1 =$ identity, $\pi_1 \sigma_2 = 0$, $\pi_2 \sigma_1 = 0$, $\pi_2 \sigma_2 =$ identity. Finally if $a \in A$, then $\sigma_1 \pi_1(a)$ is the projection of a on $\operatorname{Im} \sigma_1$ and $\sigma_2 \pi_2(a)$ is the projection of a on B. Thus

$$\sigma_1 \pi_1(a) + \sigma_2 \pi_2(a) = a \quad \text{or} \quad \sigma_1 \pi_1 + \sigma_2 \pi_2 = \text{identity}.$$

Accordingly (1) implies (4).

Definition. *Let* $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ *be an exact sequence in* $\mathscr{C}_\Lambda$. *If the four equivalent conditions of Theorem 4 hold, then it is called a 'split exact sequence'.*

We now see, in view of Theorem 3, that if $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$, then

$$0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1 \to 0$$

are split exact sequences. On the other hand, if $0 \to B \to A \to C \to 0$ is a split exact sequence, then we have an isomorphism $A \approx B \oplus C$.

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Theorem 5. *Let* $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ *be a split exact sequence in* $\mathscr{C}_\Lambda$ *and* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *an additive covariant functor. Then*

$$0 \to F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2) \to 0$$

is a split exact sequence in $\mathscr{C}_\Delta$.

Proof. Choose $\sigma_2: A_2 \to A$, $\pi_1: A \to A_1$ so that

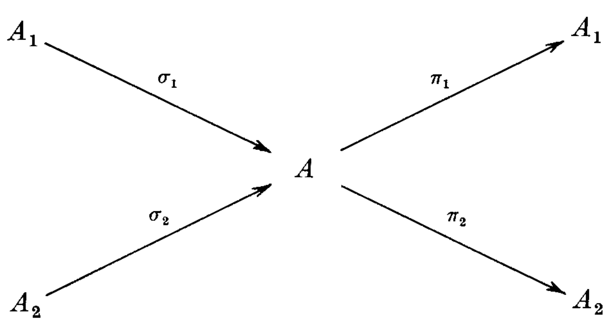
$$[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2.$$

Then, by Theorem 1, $[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)] = F(A_1) * F(A_2)$ and therefore

$$0 \to F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2) \to 0$$

is a split exact sequence by virtue of Theorem 4.

Exercise 3. *In the diagram*



suppose that $\pi_1 \sigma_1 =$ *identity and that* $\pi_2 \sigma_2 =$ *identity. Suppose also that*

$$A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \quad \textit{and} \quad A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1$$

are exact. Show that $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$.

Exercise 4. *Let* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be a covariant functor and suppose that whenever* $0 \to A' \to A \to A'' \to 0$ *is a split exact sequence in* $\mathscr{C}_\Lambda$, *then* $0 \to F(A') \to F(A) \to F(A'') \to 0$ *is a split exact sequence in* $\mathscr{C}_\Delta$. *Deduce that F is additive.*

Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a covariant functor. Assume that whenever $0 \to A' \to A \to A'' \to 0$ is exact in $\mathscr{C}_\Lambda$, then

$$0 \to F(A') \to F(A) \to F(A'')$$

resp. $F(A') \to F(A) \to F(A'') \to 0$

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is exact in $\mathscr{C}_\Delta$. In these circumstances we say that F is *left exact* resp. *right exact*. Should it be the case that the exactness of

$$0 \to A' \to A \to A'' \to 0$$

only implies that of $\quad F(A') \to F(A) \to F(A'')$,

then F is said to be *half exact*. If F is both left and right exact, i.e. if $0 \to A' \to A \to A'' \to 0$ is exact always implies that

$$0 \to F(A') \to F(A) \to F(A'') \to 0$$

is exact, then F is said to be an *exact* functor.

Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a covariant functor. If F is left exact then it preserves monomorphisms, whereas if it is right exact it preserves epimorphisms.

Lemma 2. *Suppose that the covariant functor F is left exact and that $0 \to A_1 \to A \to A_2$ is exact in $\mathscr{C}_\Lambda$. Then $0 \to F(A_1) \to F(A) \to F(A_2)$ is exact in $\mathscr{C}_\Delta$.*

The proofs of this and the next two lemmas are straightforward and will be omitted. In both Lemmas 3 and 4, F is understood to be a *covariant* functor from $\mathscr{C}_\Lambda$ to $\mathscr{C}_\Delta$.

Lemma 3. *Suppose that F is right exact and $A_1 \to A \to A_2 \to 0$ is exact in $\mathscr{C}_\Lambda$. Then $F(A_1) \to F(A) \to F(A_2) \to 0$ is exact in $\mathscr{C}_\Delta$.*

Lemma 4. *Suppose that F is exact and $A_1 \to A \to A_2$ is an exact sequence in $\mathscr{C}_\Lambda$. Then $F(A_1) \to F(A) \to F(A_2)$ is exact in $\mathscr{C}_\Delta$.*

Theorem 6. *If the covariant functor F is half exact, then it is additive.*

Proof. Let $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$ in $\mathscr{C}_\Lambda$. By Theorem 2, it is enough to show that $[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)]$ equals

$$F(A_1) * F(A_2).$$

Now by Theorem 3 and the half exactness of F,

$$F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2)$$

and
$$F(A_2) \xrightarrow{F(\sigma_2)} F(A) \xrightarrow{F(\pi_1)} F(A_1)$$

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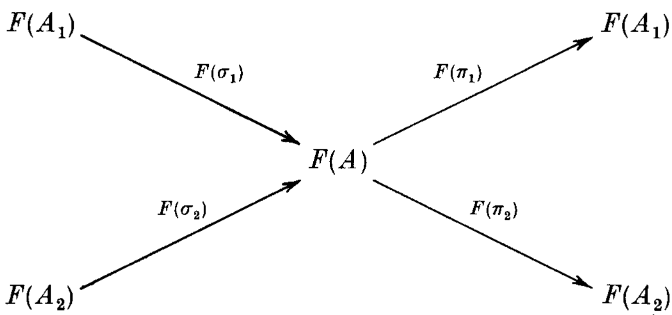
are exact. Consider the diagram



Exercise 3 shows at once that

$$[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)] = F(A_1) * F(A_2).$$

This completes the proof.

Suppose that the covariant functor F is exact and that A_1 is a submodule of the Λ-module A. Then the inclusion mapping gives rise to an exact sequence $0 \to A_1 \to A$ and therefore $0 \to F(A_1) \to F(A)$ is exact. Thus $F(A_1)$ may be regarded as a Δ-submodule of $F(A)$. This observation is relevant to the next two exercises.

Exercise 5. *Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a covariant exact functor. Show that F preserves images and kernels.*

Exercise 6. *Suppose that the functor $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ is exact and covariant, that A is a Λ-module, and $A_1, A_2, \ldots, A_n$ are Λ-submodules of A. Show that, as submodules of $F(A)$,*

$$F(A_1 + A_2 + \ldots + A_n) = F(A_1) + F(A_2) + \ldots + F(A_n)$$

and $$F(A_1 \cap A_2 \cap \ldots \cap A_n) = F(A_1) \cap F(A_2) \cap \ldots \cap F(A_n).$$

1.4 Contravariant functors

We now introduce a second type of functor. Assume that with each module A in $\mathscr{C}_\Lambda$ there is associated a module $G(A)$ in $\mathscr{C}_\Delta$ and that with each Λ-homomorphism $f: A \to A'$ there is associated a Δ-homomorphism $G(f): G(A') \to G(A)$. Assume further that

(1) $G(i_A) = i_{G(A)}$ *for all A in $\mathscr{C}_\Lambda$;*

(2) $G(gf) = G(f)\,G(g)$ *whenever $f: A \to A'$ and $g: A' \to A''$ in $\mathscr{C}_\Lambda$.*

We then say that we have a *contravariant functor* G from Λ-modules to Δ-modules.

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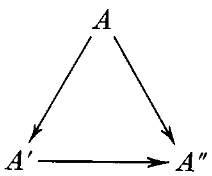
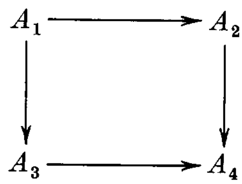
Let $G:\mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a contravariant functor. If



and



are commutative diagrams in $\mathscr{C}_\Lambda$, then applying G leaves them commutative but the arrows are reversed. Also if $f:A \to A'$ is an isomorphism and $g:A' \to A$ is its inverse, then $G(f):G(A') \to G(A)$ is an isomorphism and $G(g):G(A) \to G(A')$ is its inverse. It is clear how *additive* contravariant functors are defined. If G is additive, then it converts null homomorphisms and null objects into null homomorphisms and null objects.

Theorem 7. *Let $G:\mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be an additive contravariant functor and let $[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * \ldots * A_n$ in $\mathscr{C}_\Lambda$. Then*

$$[G(\pi_1), \ldots, G(\pi_n); G(A); G(\sigma_1), \ldots, G(\sigma_n)] = G(A_1) * G(A_2) * \ldots * G(A_n)$$

in $\mathscr{C}_\Delta$.

Proof. Apply G to the relations $\pi_j \sigma_i = \delta_{ji}$ and $\Sigma \sigma_i \pi_i =$ identity.

Theorem 8. *Let $G:\mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a contravariant functor and suppose that whenever $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$ in $\mathscr{C}_\Lambda$, then*

$$[G(\pi_1), G(\pi_2); G(A); G(\sigma_1), G(\sigma_2)] = G(A_1) * G(A_2)$$

in $\mathscr{C}_\Delta$. In these circumstances G is additive.

Proof. Suppose that $f_1, f_2: A \to B$ and let $[\sigma_1, \sigma_2; X; \pi_1, \pi_2] = B * B$. Then $\pi_1 + \pi_2: X \to B$. Let $g: A \to X$ be such that $\pi_\mu g = f_\mu$, and therefore $(\pi_1 + \pi_2) g = \pi_1 g + \pi_2 g = f_1 + f_2$. By hypothesis,

$$[G(\pi_1), G(\pi_2); G(X); G(\sigma_1), G(\sigma_2)] = G(B) * G(B).$$

Furthermore $$G(\pi_1 + \pi_2): G(B) \to G(X)$$

and $$G(\sigma_\mu)\, G(\pi_1 + \pi_2) = G(\overline{\pi_1 + \pi_2}\, \sigma_\mu) = i_{G(B)}.$$

In addition $$G(\pi_1) + G(\pi_2): G(B) \to G(X),$$

and $$G(\sigma_\mu)\, (G(\pi_1) + G(\pi_2)) = i_{G(B)}$$

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as well. It follows (see Exercise 1) that $G(\pi_1+\pi_2) = G(\pi_1)+G(\pi_2)$. Finally

$$\begin{aligned} G(f_1+f_2) &= G(\overline{\pi_1+\pi_2}g) = G(g)\,G(\pi_1+\pi_2) = G(g)\,(G(\pi_1)+G(\pi_2)) \\ &= G(g)\,G(\pi_1)+G(g)\,G(\pi_2) = G(\pi_1 g)+G(\pi_2 g) = G(f_1)+G(f_2). \end{aligned}$$

Theorem 9. *Let* $0\to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ *be a split exact sequence in* $\mathscr{C}_\Lambda$ *and let* $G:\mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be an additive contravariant functor. Then*

$$0\to G(A_2) \xrightarrow{G(\pi_2)} G(A) \xrightarrow{G(\sigma_1)} G(A_1)\to 0$$

is a split exact sequence in $\mathscr{C}_\Delta$.

Proof. This is the analogue of Theorem 5 and we simply modify the proof of that result.

Let $G:\mathscr{C}_\Lambda\to\mathscr{C}_\Delta$ be a contravariant functor. It is said to be *left exact* if whenever $0\to A'\to A\to A''\to 0$ is exact in $\mathscr{C}_\Lambda$, then

$$0\to G(A'')\to G(A)\to G(A') \tag{1.4.1}$$

is exact in $\mathscr{C}_\Delta$. We say that G is *right exact* if, in place of (1.4.1), we have an exact sequence

$$G(A'')\to G(A)\to G(A')\to 0. \tag{1.4.2}$$

Furthermore G is called *half exact* if the exactness of

$$0\to A'\to A\to A''\to 0$$

only implies that of

$$G(A'')\to G(A)\to G(A'). \tag{1.4.3}$$

Finally G is said to be *exact* if it is both left exact and right exact, i.e. if the exactness of the sequence $0\to A'\to A\to A''\to 0$ implies that

$$0\to G(A'')\to G(A)\to G(A')\to 0 \tag{1.4.4}$$

is exact.

Lemma 5. *Suppose that* $A'\to A\to A''\to 0$ *is exact in* $\mathscr{C}_\Lambda$ *and that* $G:\mathscr{C}_\Lambda\to\mathscr{C}_\Delta$ *is contravariant and left exact. Then*

$$0\to G(A'')\to G(A)\to G(A')$$

is exact in $\mathscr{C}_\Delta$.

This is trivial as are Lemmas 6 and 7 provided that one bears in mind that a left exact contravariant functor converts an epimorphism into a monomorphism whereas a right exact contravariant functor changes a monomorphism into an epimorphism.

Lemma 6. *Suppose that* $G:\mathscr{C}_\Lambda\to\mathscr{C}_\Delta$ *is contravariant and right exact. If now* $0\to A'\to A\to A''$ *is exact in* $\mathscr{C}_\Lambda$, *then* $G(A'')\to G(A)\to G(A')\to 0$ *is exact in* $\mathscr{C}_\Lambda$.

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