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K. G. Binmore
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13 DISTANCE

13.1 The space $\mathbb{R}^n$

Those readers who know a little linear algebra will find the first half of this chapter very elementary and may therefore prefer to skip forward to §13.18.

The objects in the set $\mathbb{R}^n$ are the n-tuples

$$(x_1, x_2, \ldots, x_n)$$

in which $x_1, x_2, \ldots, x_n$ are real numbers. We usually use a single symbol $\mathbf{x}$ for the n-tuple and write

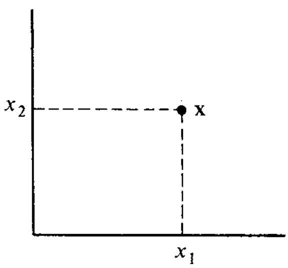
$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

The real numbers $x_1, x_2, \ldots, x_n$ are called the *co-ordinates* or the *components* of $\mathbf{x}$.

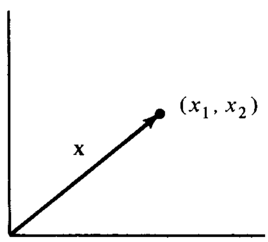
It is often convenient to refer to an object $\mathbf{x}$ in $\mathbb{R}^n$ as a *vector*. When doing so, ordinary real numbers are called *scalars*. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors and α is a scalar, we define '*vector addition*' and '*scalar multiplication*' by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

These definitions have a simple geometric interpretation which we shall illustrate in the case $n = 2$. An object $\mathbf{x} \in \mathbb{R}^2$ may be thought of as a point in the plane referred to rectangular Cartesian axes. Alternatively, we can think of $\mathbf{x}$ as an arrow with its blunt end at the origin and its sharp end at the point (x_1, x_2).



x as a point



x as an arrow

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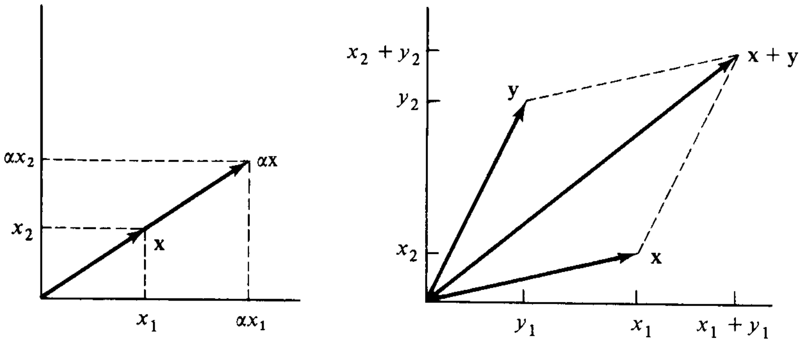
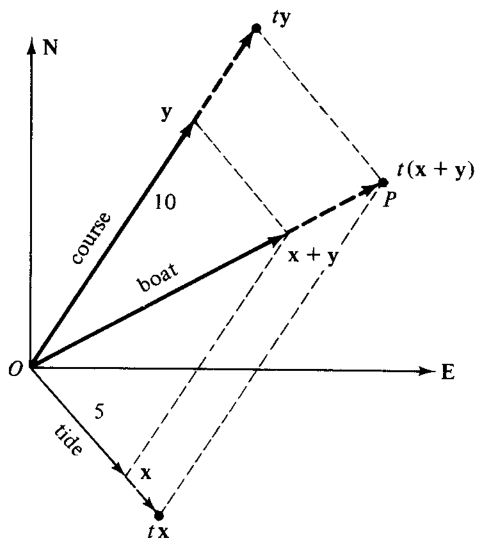
Vector addition and scalar multiplication can then be illustrated as in the diagrams below. For obvious reasons, the rule for adding two vectors is called the *parallelogram law.*



The parallelogram law is the reason that the navigators of small boats draw little parallelograms all over their charts. Suppose a boat is at O and the navigator wishes to reach point P. Assuming that the boat can proceed at 10 knots in any direction and that the tide is moving at 5 knots in a south-easterly direction, what course should be set?



The vector **x** represents that path of the boat if it drifted on the tide for an hour (distances measured in nautical miles). The vector **y** represents the path of the boat if there were no tide and it sailed the course indicated for an hour. The vector $\mathbf{x}+\mathbf{y}$ represents the path of the boat (over the sea bed) if both influences act together. The scalar t is the time it will take to reach P.

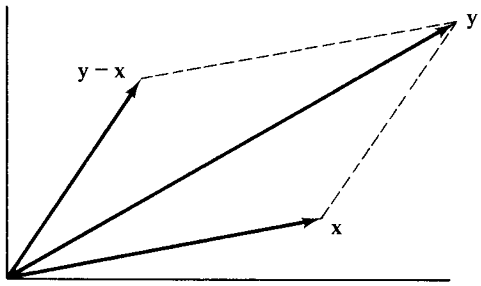
13.2 *Example* Let $\mathbf{x}=(1, 2, 3)$ and $\mathbf{y}=(2, 0, 5)$. Then

$$\mathbf{x}+\mathbf{y}=(1, 2, 3)+(2, 0, 5)=(3, 2, 8)$$
$$2\mathbf{x}=2(1, 2, 3)=(2, 4, 6).$$

It is very easy to check $\mathbb{R}^n$ is a commutative group under vector addition. (See §6.6.) This simply means that the usual rules for addition and subtraction are true. The zero vector is, of course,

$$\mathbf{0}=(0, 0,\ldots, 0).$$

The diagram below illustrates the vector $\mathbf{y}-\mathbf{x}=(y_1-x_1, y_2-x_2,\ldots, y_n-x_n)$ in the case $n=2$.



It is natural to ask about the multiplication of vectors. Is it possible to define the product of two vectors $\mathbf{x}$ and $\mathbf{y}$ as another vector $\mathbf{z}$ in a satisfactory way? There is no problem when $n=1$ since we can then identify $\mathbb{R}^1$ with $\mathbb{R}$. Nor is there a problem when $n=2$ since we can then identify $\mathbb{R}^2$ with $\mathbb{C}$ (§10.20). If $n \geqq 3$, however, there is no entirely satisfactory way of defining multiplication in $\mathbb{R}^n$. Instead we define a number of different types of 'product' none of which has all the properties which we would like a product to have.

Scalar multiplication, for example, tells us how to multiply a scalar and a vector. It does not help in multiplying two vectors. The 'inner product', which we shall meet in §13.3, tells how two vectors can be 'multiplied' to produce a scalar. In $\mathbb{R}^3$, one can introduce the 'outer product' or 'vector product' of two vectors $\mathbf{x}$ and $\mathbf{y}$. This is a vector denoted by $\mathbf{x}\wedge\mathbf{y}$ or $\mathbf{x}\times\mathbf{y}$. Unfortunately, $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$.

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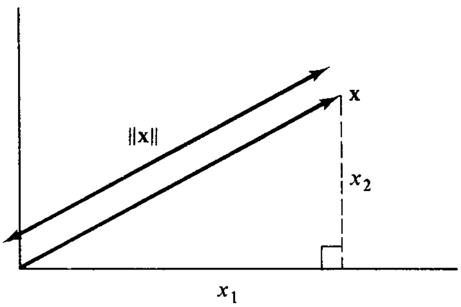
Multiplication is therefore something which does not work very well with vectors. Division is almost always *meaningless.*

13.3 Length and angle in $\mathbb{R}^n$

The *Euclidean norm* of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is defined by

$$\|\mathbf{x}\| = \{x_1^2 + x_2^2 + \ldots + x_n^2\}^{1/2}.$$

We think of $\|\mathbf{x}\|$ as the *length* of the vector $\mathbf{x}$. This interpretation is justified in $\mathbb{R}^2$ by Pythagoras' theorem (13.15).



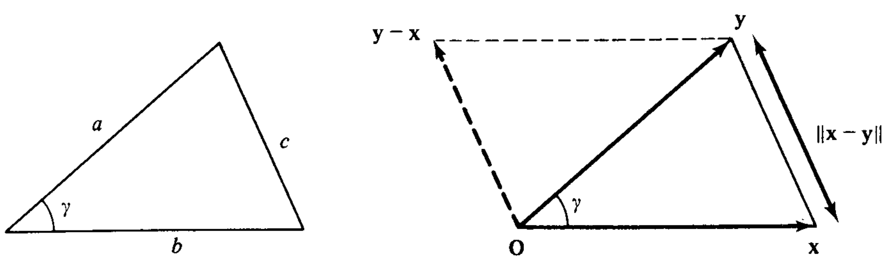
The *inner product* of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

It is easy to check the following properties:

(i) $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
(iii) $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$.

The geometric significance of the inner product can be discussed using the *cosine rule* (i.e. $c^2 = a^2 + b^2 - 2ab \cos \gamma$) in the diagram below.



Rewriting the cosine rule in terms of the vectors introduced in the righthand diagram, we obtain that

$$\|\mathbf{x}-\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\|\mathbf{x}\|\cdot\|\mathbf{y}\|\cos\gamma.$$

But,

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2&=\langle\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle=\langle\mathbf{x},\mathbf{x}-\mathbf{y}\rangle-\langle\mathbf{y},\mathbf{x}-\mathbf{y}\rangle\\&=\langle\mathbf{x},\mathbf{x}\rangle-2\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{y},\mathbf{y}\rangle=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle\mathbf{x},\mathbf{y}\rangle\end{aligned}$$

It follows that

$$\langle\mathbf{x},\mathbf{y}\rangle=\|\mathbf{x}\|\cdot\|\mathbf{y}\|\cos\gamma.$$

Of course, this argument does not *prove* anything. It simply indicates why it is helpful to think of

$$\frac{\langle\mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}$$

as the cosine of the *angle* between $\mathbf{x}$ and $\mathbf{y}$.

13.4 *Example* Find the lengths of and the cosine of the angle between the vectors $\mathbf{x}=(1,2,3)$ and $\mathbf{y}=(2,0,5)$ in $\mathbb{R}^3$.

We have that

$$\begin{aligned}\|\mathbf{x}\|&=\{1^2+2^2+3^2\}^{1/2}=\sqrt{14},\\\|\mathbf{y}\|&=\{2^2+0^2+5^2\}^{1/2}=\sqrt{29},\\\frac{\langle\mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}&=\frac{1\cdot2+2\cdot0+3\cdot5}{\sqrt{14}\sqrt{29}}=\frac{17}{\sqrt{14\times29}}.\end{aligned}$$

13.5 Some inequalities

In the previous section γ was the angle between $\mathbf{x}$ and $\mathbf{y}$. The fact that $|\cos\gamma|\leq1$ translates into the following theorem.

13.6 *Theorem* (*Cauchy–Schwarz inequality*) If $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^n$, then

$$|\langle\mathbf{x},\mathbf{y}\rangle|\leq\|\mathbf{x}\|\cdot\|\mathbf{y}\|.$$

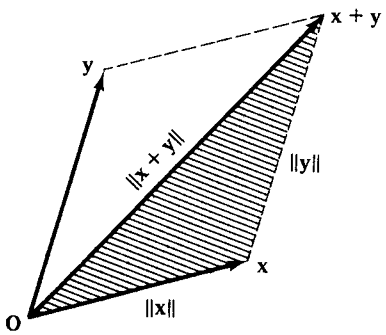
Proof Let $\alpha\in\mathbb{R}$. Then

$$\begin{aligned}0\leq\|\mathbf{x}-\alpha\mathbf{y}\|^2&=\langle\mathbf{x}-\alpha\mathbf{y},\mathbf{x}-\alpha\mathbf{y}\rangle\\&=\|\mathbf{x}\|^2-2\alpha\langle\mathbf{x},\mathbf{y}\rangle+\alpha^2\|\mathbf{y}\|^2.\end{aligned}$$

It follows that the quadratic equation $\|\mathbf{x}\|^2 - 2\alpha\langle \mathbf{x}, \mathbf{y}\rangle + \alpha^2\|\mathbf{y}\|^2$ has at most one real root (§10.10). Hence '$b^2 - 4ac \leqq 0$' – i.e.

$$4\langle \mathbf{x}, \mathbf{y}\rangle^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leqq 0.$$

It is a familiar fact in Euclidean geometry that one side of a triangle is shorter than the sum of the lengths of the other two sides.



This geometric idea translates into the following theorem.

13.7 *Theorem* (*Triangle inequality*) If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, then

$$\|\mathbf{x}+\mathbf{y}\| \leqq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle \\ &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \\ &\leqq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|.\|\mathbf{y}\| + \|\mathbf{y}\|^2 \quad \text{(theorem 13.6)} \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

13.8 *Corollary* If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, then

$$\|\mathbf{x}-\mathbf{y}\| \geqq \|\mathbf{x}\| - \|\mathbf{y}\|.$$

Proof It follows from the triangle inequality that

$$\|\mathbf{x}\| = \|(\mathbf{x}-\mathbf{y}) + \mathbf{y}\| \leqq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|.$$

13.9 Modulus

The *modulus* $|x|$ of a real number x coincides with the Euclidean norm of x thought of as a vector in $\mathbb{R}^1$. We have that

$$\|x\| = \{x^2\}^{1/2} = |x| = \begin{cases} x \ (x \geqq 0) \\ -x \ (x < 0) \end{cases}.$$

One has to remember that $y^{1/2}$ represents the *non-negative* number whose square is y (§9.13).

The modulus $|z|$ of a complex number $z = x + iy$ is identified with the Euclidean norm of (x, y) thought of as a vector in $\mathbb{R}^2$. We have that

$$|x + iy| = \{x^2 + y^2\}^{1/2} = \|(x, y)\|.$$

13.10 *Theorem* Suppose that u and v are real or complex numbers. Then

$$|uv| = |u| \cdot |v|.$$

Proof We need only consider the complex case. If $u = a + ib$ and $v = c + id$, then

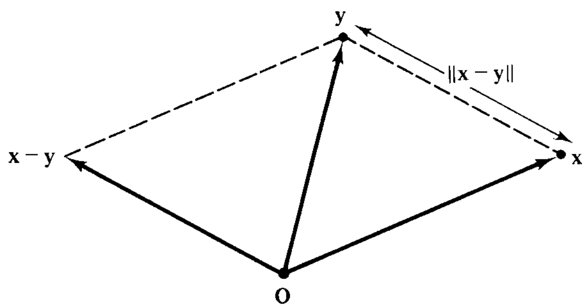
$$|uv|^2 - |u|^2|v|^2 = (ac - bd)^2 + (bc + ad)^2 - (a^2 + b^2)(c^2 + d^2) = 0.$$

13.11 Distance

The *distance* $d(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

In interpreting this idea in $\mathbb{R}^n$, it is better to think of $\mathbf{x}$ and $\mathbf{y}$ as the points at the end of the arrows rather than the arrows themselves.



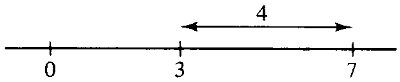
13.12 *Examples*

(i) The distance between the vectors $\mathbf{x}=(1, 2, 3)$ and $\mathbf{y}=(2, 0, 5)$ in $\mathbb{R}^3$ is

$$\|\mathbf{y}-\mathbf{x}\|=\{(2-1)^2+(0-2)^2+(5-3)^2\}^{1/2}=3.$$

(ii) The distance between 3 and 7 (regarded as vectors in $\mathbb{R}^1$) is

$$|3-7|=|-4|=4.$$



13.13 *Exercise*

(1) Let $\mathbf{x}=(0, 1, 0)$ and $\mathbf{y}=(1, 1, 0)$ be vectors in $\mathbb{R}^3$. Calculate the quantities

(i) $\mathbf{x}+\mathbf{y}$ (ii) $\mathbf{x}-\mathbf{y}$ (iii) $2\mathbf{x}$
(iv) $\|\mathbf{x}\|$ (v) $\|\mathbf{x}-\mathbf{y}\|$ (vi) $\langle \mathbf{x}, \mathbf{y}\rangle$.

What is the length of the vector $\mathbf{x}$? What are the distance and the angle between $\mathbf{x}$ and $\mathbf{y}$?

(2) Calculate the moduli of the following real and complex numbers:
(i) -3 (ii) 0 (iii) 4 (iv) $3+4i$ (v) $4-3i$ (vi) i.

(3) If a and b are any real numbers, prove that $|a|<b$ if and only if $-b<a<b$. If $|a|<b$ for *all* $b>0$, prove that $a=0$.

(4) Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Prove that

$$d(\mathbf{x}, \mathbf{y}) \geqq |\ \|\mathbf{x}\|-\|\mathbf{y}\|\ |.$$

[*Hint*: Use question 3.]

(5) Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Prove that

$$\langle \mathbf{x}, \mathbf{y}\rangle=\tfrac{1}{4}\{\|\mathbf{x}+\mathbf{y}\|^2-\|\mathbf{x}-\mathbf{y}\|^2\}.$$

(6) Let $\mathbf{x}\in\mathbb{R}^m$ and $\mathbf{y}\in\mathbb{R}^n$. Then $(\mathbf{x}, \mathbf{y})$ may be regarded as an element of $\mathbb{R}^{m+n}$. Prove that

$$\|(\mathbf{x}, \mathbf{y})\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2.$$

13.14 Euclidean geometry and $\mathbb{R}^n$

In §10.1, we explained that it is the models in terms of which a system of axioms can be interpreted which make that system interesting. But this is a two-way process. It is equally true that a model is interesting because of the systems of axioms which it may satisfy.

The space $\mathbb{R}^2$ is particularly interesting because it serves as a model for the axioms of Euclidean geometry in the plane. The axioms given by Euclid himself are incomplete by modern standards in that certain of his theorems cannot be deduced from his axioms alone but depend also on implicit geometrical assumptions which are systematically taken for granted but never explicitly stated. (Some of these are mentioned briefly later on.) When we refer to the axioms of Euclidean geometry, we mean the axiom system given by David Hilbert in his famous book *The foundations of geometry* published at the beginning of this century.

The geometric terms which appear in Hilbert's axioms are the words *point*, *line*, *lie on*, *between* and *congruent*. To show that $\mathbb{R}^2$ is a model for Euclidean plane geometry one has to give a precise definition of each of these words in terms of $\mathbb{R}^2$ and then prove each of Hilbert's axioms for Euclidean plane geometry as a theorem in $\mathbb{R}^2$. This will demonstrate, in particular, that the axioms for Euclidean geometry are *consistent*.

An attempt at describing this programme in detail is beyond the scope of this book. (Interested readers will find the book *Elementary geometry from an advanced standpoint* by E. E. Moise (Addison-Wesley, 1963) an excellent reference.) Instead, we shall merely explain how some of the very familiar ideas of elementary geometry are expressed in terms of the space $\mathbb{R}^n$.

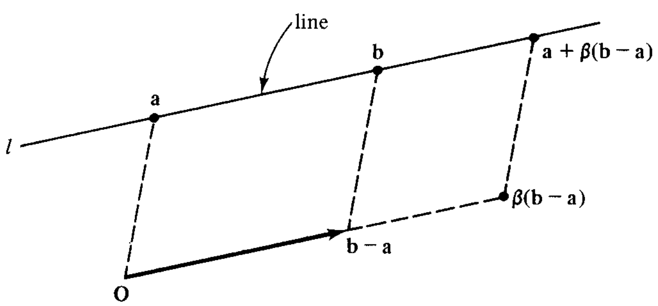
A *point* is simply an element $\mathbf{x}\in\mathbb{R}^n$. The *line* l through the distinct points $\mathbf{a}$ and $\mathbf{b}$ is the set

$$l=\{\alpha\mathbf{a}+\beta\mathbf{b}:\alpha\in\mathbb{R},\ \beta\in\mathbb{R}\text{ and }\alpha+\beta=1\}.$$

This is easier to illustrate with a picture if we rewrite it in the form

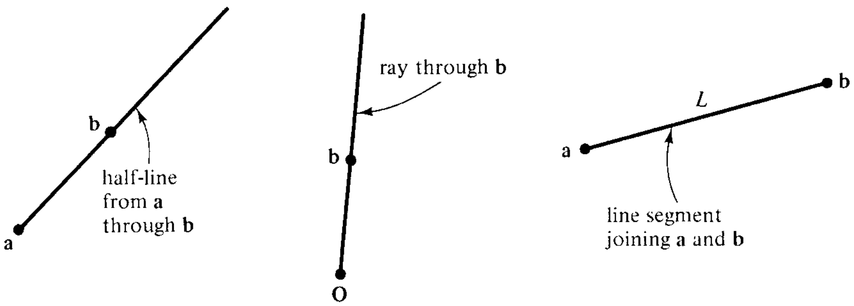
$$l=\{\mathbf{a}+\beta(\mathbf{b}-\mathbf{a}):\beta\in\mathbb{R}\}$$

and think of l as the line through the point $\mathbf{a}$ in the direction of the vector $\mathbf{b}-\mathbf{a}$.



If, in the above, β is restricted to be *non-negative* we obtain the definition of the *half-line* from $\mathbf{a}$ through $\mathbf{b}$. If $\mathbf{a}=\mathbf{0}$, such a half-line is called a *ray*. If both α and β are restricted to be non-negative we obtain the definition of a line segment. Thus the *line segment* L joining $\mathbf{a}$ and $\mathbf{b}$ is given by

$$L=\{\alpha\mathbf{a}+\beta\mathbf{b}: \alpha\geqq 0, \beta\geqq 0 \text{ and } \alpha+\beta=1\}.$$



Some authors do not insist that the endpoints of a line segment belong to the line segment. Where it matters whether or not the endpoints of a line segment L belong to L we shall therefore sometimes stress the fact that they do by calling L a *closed* line segment.

We have already discussed how *length* and *angle* are introduced into $\mathbb{R}^n$ via the norm and the inner product of the space. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* if and only if

$$\langle\mathbf{x}, \mathbf{y}\rangle=0.$$

Like most important ideas in mathematics, the word orthogonal has numerous synonyms of which some are *perpendicular*, *normal* and '*at right angles*'.

13.15 (*Pythagoras' theorem*) Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if

$$\|\mathbf{x}+\mathbf{y}\|^2=\|\mathbf{x}\|^2+\|\mathbf{y}\|^2.$$