Geometrical methods of mathematical physics

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PREFACE

Why study geometry?

This book aims to introduce the beginning or working physicist to a wide range of analytic tools which have their origin in differential geometry and which have recently found increasing use in theoretical physics. It is not uncommon today for a physicist's mathematical education to ignore all but the simplest geometrical ideas, despite the fact that young physicists are encouraged to develop mental 'pictures' and 'intuition' appropriate to physical phenomena. This curious neglect of 'pictures' of one's mathematical tools may be seen as the outcome of a gradual evolution over many centuries. Geometry was certainly extremely important to ancient and medieval natural philosophers; it was in geometrical terms that Ptolemy, Copernicus, Kepler, and Galileo all expressed their thinking. But when Descartes introduced coordinates into Euclidean geometry, he showed that the study of geometry could be regarded as an application of algrebra. Since then, the importance of the study of geometry in the education of scientists has steadily declined, so that at present a university undergraduate physicist or applied mathematician is not likely to encounter much geometry at all.

One reason for this suggests itself immediately: the relatively simple geometry of the three-dimensional Euclidean world that the nineteenth-century physicist believed he lived in can be mastered quickly, while learning the great diversity of analytic techniques that must be used to solve the differential equations of physics makes very heavy demands on the student's time. Another reason must surely be that these analytic techniques were developed at least partly in response to the profound realization by physicists that the laws of nature could be expressed as differential equations, and this led most mathematical physicists genuinely to neglect geometry until relatively recently.

However, two developments in this century have markedly altered the balance between geometry and analysis in the twentieth-century physicist's outloook. The first is the development of the theory of relativity, according to which the Euclidean three-space of the nineteenth-century physicist is only an approximation to the correct description of the physical world. The second development, which is only beginning to have an impact, is the realization by twentieth-century mathematicians, led by Cartan, that the relation between geometry and analysis is a two-way street: on the one hand analysis may be the foundation of the study of geometry, but on the other hand the study of geometry leads naturally to the development of certain analytic tools (such as the Lie derivative and the exterior calculus) and certain concepts (such as the manifold, the fiber bundle, and the identification of vectors with derivatives) that have great power in applications of analysis. In the modern view, geometry remains subsidiary to analysis. For example, the basic concept of differential geometry, the differentiable manifold, is defined in terms of real numbers and differentiable functions. But this is no disadvantage: it means that concepts from analysis can be expressed geometrically, and this has considerable heuristic power.

Because it has developed this intimate connection between geometrical and analytic ideas, modern differential geometry has become more and more important in theoretical physics, where it has led to a greater simplicity in the mathematics and a more fundamental understanding of the physics. This revolution has affected not only special and general relativity, the two theories whose content is most obviously geometrical, but other fields where the geometry involved is not always that of physical space but rather of a more abstract space of variables: electromagnetism, thermodynamics, Hamiltonian theory, fluid dynamics, and elementary particle physics.

Aims of this book

In this book I want to introduce the reader to some of the more important notions of twentieth-century differential geometry, trying always to use that geometrical or 'pictorial' way of thinking that is usually so helpful in developing a physicist's intuition. The book attempts to teach mathematics, not physics. I have tried to include a wide range of applications of this mathematics to branches of physics which are familiar to most advanced undergraduates. I hope these examples will do more than illustrate the mathematics: the new mathematical formulation of familiar ideas will, if I have been successful, give the reader a deeper understanding of the physics.

I will discuss the background I have assumed of the reader in more detail below, but here it may be helpful to give a brief list of some of the 'familiar' ideas which are seen in a new light in this book: vectors, tensors, inner products, special relativity, spherical harmonics and the rotation group (and angular-momentum operators), conservation laws, volumes, theory of integration, curl and cross-product, determinants of matrices, partial differential equations and their integrability conditions, Gauss' and Stokes' integral theorems of vector calculus, thermodynamics of simple systems, Caratheodory's theorem (and the second law of thermodynamics), Hamiltonian systems in phase space, Maxwell's equations, fluid dynamics (including the laws governing the conservation of circulation), vector calculus in curvilinear coordinate systems, and the quantum theory of a charged scalar field. Besides these more or less familiar subjects, there are a few others which are not usually taught at undergraduate level but which most readers would certainly have heard of: the theory of Lie groups and symmetry, open and closed cosmologies, Riemannian geometry, and gauge theories of physics. That all of these subjects can be studied by the methods of differential geometry is an indication of the importance differential geometry is likely to have in theoretical physics in the future.

I believe it is important for the reader to develop a pictorial way of thinking and a feeling for the 'naturalness' of certain geometrical tools in certain situations. To this end I emphasize repeatedly the idea that tensors are geometrical objects, defined independently of any coordinate system. The role played by components and coordinate transformations is submerged into a secondary position: whenever possible I write equations without indices, to emphasize the coordinate-independence of the operations. I have made no attempt to present the material in a strictly rigorous or axiomatic way, and I have had to ignore many aspects of our subject which a mathematician would regard as fundamental. I do, of course, give proofs of all but a handful of the most important results (references for the exceptions are provided), but I have tried wherever possible to make the main geometrical ideas in the proof stand out clearly from the background of manipulation. I want to show the beauty, elegance, and naturalness of the mathematics with the minimum of obscuration.

How to use this book

The first chapter contains a review of the sort of elementary mathematics assumed of the reader plus a short introduction to some concepts, particularly in topology, which undergraduates may not be familiar with. The next chapters are the core of the book: they introduce tensors, Lie derivatives, and differential forms. Scattered through these chapters are some applications, but most of the physical applications are left for systematic treatment in chapter 5. The final chapter, on Riemannian geometry, is more advanced and makes contact with areas of particle physics and general relativity in which differential geometry is an everyday tool.

The material in this book should be suitable for a one-term course, provided the lecturer exercises some selection in the most difficult areas. It should also be possible to teach the most important points as a unit of, say, ten lectures in an advanced course on mathematical methods. I have taught such a unit to graduate students, concentrating mainly on §§ 2.1–2.3, 2.5–2.8, 2.12–2.14, 2.16, 2.17, 2.19–2.28, 3.1–3.13, 4.1–4.6, 4.8, 4.14–4.18, 4.20–4.23, 4.25, 4.26, 5.1, 5.2, 5.4–5.7, and 5.15–5.18. I hope lecturers will experiment with their own choices of material, especially because there are many people for whom geometrical reasoning is easier and more natural than purely analytic reasoning, and for them an early exposure to geometrical ideas can only be helpful. As a general guide to selecting material, section headings within chapters are printed in two different styles. Fundamental material is marked by **boldface** headings, while more advanced or supplementary topics are marked by ***boldface italics***. All of the last chapter falls into this category. The same convention of type-face distinguishes those exercises which are central to the development of the mathematics from those which are peripheral.

The exercises form an integral part of the book. They are inserted in the middle of the text, and they are designed to be worked when they are first encountered. Usually the text after an exercise will assume that the reader has worked and understood the exercise. The reader who does not have the time to work an exercise should nevertheless read it and try to understand its result. Hints and some solutions will be found at the end of the book.

Background assumed of the reader

Most of this book should be understandable to an advanced undergraduate or beginning graduate student in theoretical physics or applied mathematics. It presupposes reasonable facility with vector calculus, calculus of many variables, matrix algebra (including eigenvectors and determinants), and a little operator theory of the sort one learns in elementary quantum mechanics. The physical applications are drawn from a variety of fields, and not everyone will feel at home with them all. It should be possible to skip many sections on physics without undue loss of continuity, but it would probably be unrealistic to attempt this book without some familiarity with classical mechanics, special relativity, and electromagnetism. The bibliography at the end of chapter 1 lists some books which provide suitable background.

I want to acknowledge my debt to the many people, both colleagues and teachers, who have helped me to appreciate the beauty of differential geometry and understand its usefulness in physics. I am especially indebted to Kip Thorne, Rafael Sorkin, John Friedman, and Frank Estabrook. I also want to thank the first two and many patient students at University College, Cardiff, for their comments on earlier versions of this book. Two of my students, Neil Comins and Brian Wade, deserve special mention for their careful and constructive suggestions. It is also a pleasure to thank Suzanne Ball, Jane Owen, and Margaret Wilkinson for their fast and accurate typing of the manuscript through all its revisions. Finally, I thank my wife for her patience and encouragement, particularly during the last few hectic months.

Cardiff, 30 June 1979 Bernard Schutz