INTRODUCTION TO PROBABILITY AND STATISTICS

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FROM A BAYESIAN VIEWPOINT

PART 2
INFERENCE

BY

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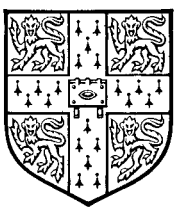

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To

M. P. MESHENBERG

in gratitude

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PREFACE

The content of the two parts of this book is the minimum that, in my view, any mathematician ought to know about random phenomena—probability and statistics. The first part deals with probability, the deductive aspect of randomness. The second part is devoted to statistics, the inferential side of our subject.

The book is intended for students of mathematics at a university. The mathematical prerequisite is a sound knowledge of calculus, plus familiarity with the algebra of vectors and matrices. The temptation to assume a knowledge of measure theory and general integration has been resisted and, for example, the concept of a Borel field is not used. The treatment would have been better had these ideas been used, but against this, the number of students able to study random phenomena by means of the book would have been substantially reduced. In any case the intent is only to provide an introduction to the subject, and at that level the measure theory concepts do not appreciably assist the understanding. A statistical specialist should, of course, continue his study further; but only, in my view, at a postgraduate level with the prerequisite of an honours degree in pure mathematics, when he will necessarily know the appropriate measure theory.

A similar approach has been adopted in the level of the proofs offered. Where a rigorous proof is available at this level, I have tried to give it. Otherwise the proof has been omitted (for example, the convergence theorem for characteristic functions) or a proof that omits certain points of refinement has been given, with a clear indication of the presence of gaps (for example, the limiting properties of maximum likelihood). Probability and statistics are branches of applied mathematics—in the proper sense of that term, and not in the narrow meaning that is common, where it means only applications to physics. This being so, some slight indulgence in the nature of the rigour is perhaps permissible. The applied nature of the subject means that the student using this book needs to supplement it with

some experience of practical data handling. No attempt has been made to provide such experience in the present book, because it would have made the book too large, and in any case other books that do provide it are readily available. The student should be trained in the use of various computers and be given exercises in the handling of data. In this way he will obtain the necessary understanding of the practical stimuli that have led to the mathematics, and the use of the mathematical results in understanding the numerical data. These two aspects of the subject, the mathematical and the practical, are complementary, and both are necessary for a full understanding of our subject. The fact that only one aspect is fully discussed here ought not to lead to neglect of the other.

The book is divided into eight chapters, and each chapter into six sections. Equations and theorems are numbered in the decimal notation: thus equation 3.5.1 refers to equation 1 of section 5 of chapter 3. Within §3.5 it would be referred to simply as equation (1). Each section begins with a formal list of definitions, with statements and proofs of theorems. This is followed by discussion of these, examples and other illustrative material. In the discussion an attempt has been made to go beyond the usual limits of a formal treatise and to place the ideas in their proper contexts; and to emphasize ideas that are of wide use as distinct from those of only immediate value. At the end of each chapter there is a large set of exercises, some of which are easy, but many of which are difficult. Most of these have been taken from examination papers, and I am grateful for permission from the Universities of London, Cambridge, Aberdeen, Wales, Manchester and Leicester to use the questions in this way. (In order to fit into the Bayesian framework some minor alterations of language have had to be made in these questions. But otherwise they have been left as originally set.)

The second part of the book, the last four chapters, 5 to 8, is devoted to statistics or inference. The first three chapters of the first part are a necessary prerequisite. Much of this part has been written in draft twice: once in an orthodox way with the use only of frequency probabilities; once in terms of probability as a degree of belief. The former treatment seemed to have so

many unsatisfactory features, and to be so difficult to present to students because of the mental juggling that is necessary in order to understand the concepts, that it was abandoned. This is not the place to criticize in detail the defects of the purely frequentist approach. Some comments have been offered in the text (§5.6, for example). Here we merely cite as an example the concept of a confidence interval in the usual sense. Technically the confidence level is the long-run coverage of the true value by the interval. In practice this is rarely understood, and is typically regarded as a degree of belief. In the approach adopted here it is so regarded, both within the formal mathematics, and practically. We use the adjective *Bayesian* to describe an approach which is based on repeated uses of Bayes's theorem.

In chapter 5 inference problems for the normal distribution are discussed. The use of Bayes's theorem to modify prior beliefs into posterior beliefs by means of the data is explained, and the important idea of vague prior knowledge discussed. These ideas are extended in chapter 6 to several normal distributions leading as far as elementary analysis of variance. In chapter 7 inferences for other distributions besides the normal are discussed: in particular goodness-of-fit tests and maximum likelihood ideas are introduced. Chapter 8 deals with least squares, particularly with tests and estimation for linear hypotheses. The intention has been to provide a sound basis consisting of the most important inferential concepts. On this basis a student should be able to apply these ideas to more specialised topics in statistics: for example, analysis of more complicated experimental designs and sampling schemes.

The main difficulty in adopting, in a text-book, a new approach to a subject (as the Bayesian is currently new to statistics) lies in adapting the new ideas to current practice. For example, hypothesis testing looms large in standard statistical practice, yet scarcely appears as such in the Bayesian literature. An unbiased estimate is hardly needed in connexion with degrees of belief. A second difficulty lies in the fact that there is no accepted Bayesian school. The approach is too recent for the mould to have set. (This has the advantage that the student can be free to think for himself.) What I have done in this book is to

develop a method which uses degrees of belief and Bayes's theorem, but which includes most of the important orthodox statistical ideas within it. My Bayesian friends contend that I have gone too far in this: they are probably right. But, to give an example, I have included an account of significance testing within the Bayesian framework that agrees excellently, in practice, with the orthodox formulation. Most of modern statistics is perfectly sound in practice; it is done for the wrong reason. Intuition has saved the statistician from error. My contention is that the Bayesian method justifies what he has been doing and develops new methods that the orthodox approach lacks. The current shift in emphasis from significance testing to interval estimation within orthodox statistics makes sense to a Bayesian because the interval provides a better description of the posterior distribution.

In interpreting classical ideas in the Bayesian framework I have used the classical terminology. Thus I have used the phrase *confidence interval* for an interval of the posterior distribution. The first time it is introduced it is called a Bayesian confidence interval, but later the first adjective is dropped. I hope this will not cause trouble. I could have used another term, such as posterior interval, but the original term is apposite and, in almost all applications, the two intervals, Bayesian and orthodox, agree, either exactly or to a good approximation. It therefore seemed foolish to introduce a second term for something which, in practice, is scarcely distinguishable from the original.

There is nothing on decision theory, apart from a brief explanation of what it is in §5.6. My task has been merely to discuss the way in which data influence beliefs, in the form of the posterior distribution, and not to explain how the beliefs can be used in decision making. One has to stop somewhere. But it is undoubtedly true that the main flowering of the Bayesian method over the next few years will be in decision theory. The ideas in this book should be useful in this development, and, in any case, the same experimental results are typically used in many different decision-making situations so that the posterior distribution is a common element to them all.

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D. V. L.

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