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Sets and groups

In this chapter we give some of the basic definitions and results of set theory and group theory that are used in the book. It is best to refer back to this chapter when the need arises.

For sets X, Y we use the notation $Y \subseteq X$ to mean that Y is a subset of X and $Y \subset X$ to mean that Y is a subset of X and $Y \neq X$. If $Y \subseteq X$ then we denote by $X - Y$ the set of the elements of X which do not belong to Y. The empty set is denoted by $\emptyset$.

The *cartesian* or *direct* product of two sets X and Y is the set of ordered pairs of the form (x, y) where $x \in X$ and $y \in Y$, i.e.

$$X \times Y = \{ (x, y); x \in X, y \in Y \} .$$

The cartesian product of a finite collection $\{ X_i; i=1, 2, ..., n \}$ of sets can be defined analogously:

$$X_1 \times X_2 \times ... \times X_n = \{ (x_1, x_2, ..., x_n); x_i \in X_i, 1 \leq i \leq n \} .$$

A *function* or *map* $f: X \to Y$ between two sets is a correspondence that associates with each element x of X a unique element f(x) of Y. The *identity function* on a set X is the function $1: X \to X$ such that $1(x) = x$ for all $x \in X$. The *image* of the function $f: X \to Y$ is defined by

$$\mathrm{Im}(f) = f(X) = \{ y \in Y; y = f(x) \text{ for some } x \in X \} .$$

Note that if W, W′ are two subsets of X then

$$f(W \cup W') = f(W) \cup f(W'),$$
$$f(W \cap W') \subseteq f(W) \cap f(W').$$

More generally, if we have a collection of subsets of X, say $\{ W_j; j \in J \}$ where J is some *indexing* set, then

$$f(\bigcup_{j \in J} W_j) = \bigcup_{j \in J} f(W_j),$$
$$f(\bigcap_{j \in J} W_j) \subseteq \bigcap_{j \in J} f(W_j).$$

We often abbreviate $f\colon X \to Y$ simply by f if no confusion can arise. A function $f\colon X \to Y$ defines a function from X to $f(X)$ which is also denoted by f. If A is a subset of X then f *restricted* to A is denoted by $f|A$; it is the function $f|A : A \to Y$ defined by $(f|A)(a) = f(a)$ for $a \in A$.

If Z is a subset of Y and $f\colon X \to Y$ is a function then the *inverse image* of Z under f is

$$f^{-1}(Z) = \{ x \in X; f(x) \in Z \}.$$

Note that

$$f^{-1}(\bigcup_{j\in J} Z_j) = \bigcup_{j\in J} f^{-1}(Z_j)$$
$$f^{-1}(\bigcap_{j\in J} Z_j) = \bigcap_{j\in J} f^{-1}(Z_j),$$
$$f^{-1}(Y-Z_j) = X-f^{-1}(Z_j),$$

for a collection $\{ Z_j; j \in J \}$ of subsets Z_j of Y.

A function $f\colon X \to Y$ is *one-to-one* or *injective* if whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. A function $f\colon X \to Y$ is *onto* or *surjective* if $f(X) = Y$. A function $f\colon X \to Y$ that is both injective and surjective is said to be *bijective*. In this case there is an *inverse function* $f^{-1}\colon Y \to X$ defined by

$$x = f^{-1}(y) \Leftrightarrow y = f(x).$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions then the *composite function* $gf\colon X \to Z$ is defined by

$$gf(x) = g(f(x)), x \in X.$$

If $f\colon X \to Y$ is a bijective function then $ff^{-1}\colon Y \to Y$ and $f^{-1}f\colon X \to X$ are the identity functions. Conversely if $gf\colon X \to X$ and $fg\colon Y \to Y$ are the identity functions then f and g are bijective functions, each being the inverse of the other. The condition that $gf\colon X \to X$ is the identity function implies that f is injective and g is surjective.

A *relation* on a set X is a subset $\sim$ of $X \times X$. We usually write $x \sim y$ if $(x,y) \in \sim$. A relation $\sim$ on X is an *equivalence relation* if it satisfies the following three conditions.

(i) The reflexive condition: $x \sim x$ for all $x \in X$.
(ii) The symmetric condition: If $x \sim y$ then $y \sim x$.
(iii) The transitive condition: If $x \sim y$ and $y \sim z$ then $x \sim z$.

The *equivalence class* of x is the set

$$[x] = \{ y \in X; x \sim y \}.$$

If $\sim$ is an equivalence relation on X then each element of X belongs to precisely one equivalence class.

A *binary operation* on a set X is a function f: X × X → X. We abbreviate f(x,y) to xy (multiplicative notation) or occasionally x + y (additive notation).

A *group* is a set G together with a binary operation satisfying three conditions:

(1) There exists an element $1 \in G$, the *identity element* of G, such that $g1 = 1g = g$ for all $g \in G$.

(2) For each $g \in G$ there is an element $g^{-1} \in G$, the *inverse* of g, such that $gg^{-1} = g^{-1}g = 1$.

(3) For all $g_1, g_2, g_3 \in G$ *associativity* holds, i.e.

$$(g_1 g_2) g_3 = g_1 (g_2 g_3).$$

In the additive group notation the identity element is denoted by 0 and the inverse of g by −g. A group whose only element is the identity is the *trivial group* $\{1\}$ or $\{0\}$.

A subset H of a group is a *subgroup* of G if H is a group under the binary operation of G. If H is a subgroup of G and $g \in G$ then the *left coset of* H *by* g is the subset

$$gH = \{ gh; h \in H \}.$$

Right cosets are defined analogously. Two left cosets gH, g′H of a subgroup H are either disjoint or identical.

The *direct product* G × H of groups G and H is the set G × H with binary operation defined by $(g,h)(g',h') = (gg',hh')$. In the additive case we refer to the *direct sum* and denote it by $G \oplus H$.

A *homomorphism* f: G → H from a group G to a group H is a function such that

$$f(gg') = f(g)f(g')$$

for all $g,g' \in G$. If the homomorphism f: G → H is bijective then we say that G and H are *isomorphic groups*, that f is an *isomorphism* and we write $G \cong H$ or $f: G \cong H$. The *kernel* of a homomorphism f: G → H is the set

$$\ker f = \{ g \in G; f(g) = 1_H \}$$

where 1_H is the identity of H. The kernel of an isomorphism consists of only the identity element of G.

A subgroup K of a group G is *normal* if $gkg^{-1} \in K$ for all $g \in G$, $k \in K$. The kernel of a homomorphism f: G → H is a normal subgroup of G. A homomorphism f: G → H is injective if and only if $\ker f = \{1\}$.

If K is a normal subgroup of G then the left coset gK equals the right coset Kg and the set G/K of all left cosets of K is a group under the operation

$$(gK)(g'K) = (gg')K.$$

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More information

We call G/K the *quotient group* of G by K.

The *first isomorphism theorem* states that if f: G → H is a surjective homomorphism from a group G to a group H with kernel K then H is isomorphic to the quotient group G/K.

If g ∈ G then the *subgroup generated by g* is the subset of G consisting of all integral powers of g

$$\langle g \rangle = \{ g^n ; n \in \mathbb{Z} \}$$

where $g^n = \overbrace{gg\ldots g}^{n}$ if $n \geq 0$ and $g^n = \overbrace{g^{-1}g^{-1}\ldots g^{-1}}^{-n}$ if $n \leq 0$. In the case of additive notation we have

$$\langle g \rangle = \{ ng; n \in \mathbb{Z} \}$$

where $ng = \overbrace{g+g+\ldots+g}^{n}$ if $n \geq 0$ and $ng = \overbrace{-g+(-g)+\ldots+(-g)}^{-n}$ if $n \leq 0$. If $G = \langle g \rangle$ for some g then we say that G is a *cyclic group* with generator g. In general a *set of generators* for a group G is a subset S of G such that each element of G is a product of powers of elements taken from S. If S is finite then we say that G is *finitely generated*.

A group G is said to be *abelian* or *commutative* if $gg' = g'g$ for all $g,g' \in G$. For example, the set of integers $\mathbb{Z}$ is an abelian group (additive notation); moreover it is a cyclic group generated by +1 or −1.

A *free abelian group of rank n* is a group isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (n copies).

The *decomposition theorem for finitely generated abelian groups* states: If G is a finitely generated abelian group then G is isomorphic to

$$H_0 \oplus H_1 \oplus \ldots \oplus H_m$$

where H_0 is a free abelian group and the H_i, i=1,2,...,m, are cyclic groups of prime power order. The rank of H_0 and the orders of the cyclic subgroups $H_1, H_2, \ldots, H_m$ are uniquely determined.

A commutator in a group G is an element of the form $ghg^{-1}h^{-1}$. The *commutator subgroup* of G is the subset of G consisting of all finite products of commutators of G (it is a subgroup). The commutator subgroup K is a normal subgroup of G and it is in fact the smallest subgroup of G for which G/K is abelian.

We use $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{Q}$ to denote the set of real numbers, complex numbers, integers, natural numbers (or positive integers) and rational numbers respectively. We often refer to $\mathbb{R}$ as the real line and to $\mathbb{C}$ as the complex plane. The set $\mathbb{R}^n$ is the cartesian product of n copies of $\mathbb{R}$. We use the following notation for certain subsets of $\mathbb{R}$ (called *intervals*):

$$(a,b) = \{ x \in \mathbb{R} ; a < x < b \},$$
$$[a,b] = \{ x \in \mathbb{R} ; a \leq x \leq b \},$$
$$[a,b) = \{ x \in \mathbb{R} ; a \leq x < b \},$$
$$(a,b] = \{ x \in \mathbb{R} ; a < x \leq b \}.$$

The meaning of the subsets $(-\infty,b)$, $(-\infty,b]$, $[a,\infty)$ and (a,∞) should be apparent. Observe that $(-\infty,\infty) = \mathbb{R}$.

Note that (a,b) could refer to a pair of elements, say in $\mathbb{R}^2$ for example, as well as an interval in $\mathbb{R}$. What is meant in a particular instance should be clear from the context.

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Background: metric spaces

In topology we study sets with some 'structure' associated with them which enable us to make sense of the question Is $f: X \to Y$ continuous or not?, where $f: X \to Y$ is a function between two such sets. In this chapter we shall find out what this 'structure' is by looking at euclidean and metric spaces.

Recall that for a function $f: \mathbb{R} \to \mathbb{R}$ we say that f is *continuous at x* if for all $\epsilon_x > 0$ there exists $\delta_x > 0$ such that $|f(y) - f(x)| < \epsilon_x$ whenever $|y - x| < \delta_x$. The function is then said to be *continuous* if it is continuous at all points $x \in \mathbb{R}$. We can extend this definition of continuity to functions $f: \mathbb{R}^n \to \mathbb{R}^m$ simply by replacing the modulus sign by the euclidean distance. More generally, if we have sets with 'distance functions' then we can define continuity using these distance functions. A 'distance function' – properly called a metric – has to satisfy some (obvious) conditions and these lead to a definition.

1.1 Definition

Let A be a set. A function $d: A \times A \to \mathbb{R}$ satisfying

(i) $d(a,b) = 0$ if and only if $a = b$,

(ii) $d(a,b) + d(a,c) \geq d(b,c)$ for all $a,b,c \in A$

is called a *metric* for A. A set A with a particular metric on it is called a *metric space* and is denoted by (A,d) or simply M.

The second property is known as the *triangle inequality*.

1.2 Exercise

Show that if d is a metric for A then $d(a,b) \geq 0$ and $d(a,b) = d(b,a)$ for all $a,b \in A$.

If we take $A = \mathbb{R}$ and $d(x,y) = |x - y|$ then it is not difficult to see that d is a metric. More generally take $A = \mathbb{R}^n$ and define d by

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = \|x-y\|$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Again, it is not hard to show that d is a metric for $\mathbb{R}^n$. This metric is called the *euclidean* or *usual* metric.

Two other examples of metrics on $A = \mathbb{R}^n$ are given by

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

We leave it as an exercise for the reader to check that these do in fact define a metric.

Finally, if A is any set then we can define a metric on it by the rules $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$. The resulting metric is called the *discrete metric* on A.

1.3 Exercises

(a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = \|x-y\| ; \quad d(x,y) = \begin{cases} 0 \text{ if } x=y, \\ 1 \text{ if } x \neq y; \end{cases}$$

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i| ; \quad d(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

(b) Show that $d(x,y) = (x - y)^2$ does not define a metric on $\mathbb{R}$.

(c) Show that $d(x,y) = \min_{1 \leq i \leq n} |x_i - y_i|$ does not define a metric on $\mathbb{R}^n$.

(d) Let d be a metric and let r be a positive real number. Show that d_r defined by $d_r(x,y) = r\,d(x,y)$ is also a metric.

(e) Let d be a metric. Show that d' defined by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric.

(f) In $\mathbb{R}^2$ define $d(x,y)$ = smallest integer greater or equal to usual distance between x and y. Is d a metric for $\mathbb{R}^2$?

Continuity between metric spaces, as we have indicated, is now easy to define.

1.4 Definition

Let (A,d_A), (B,d_B) be metric spaces. A function $f\colon A \to B$ is said to be *continuous* at $x \in A$ if and only if for all $\epsilon_x > 0$ there exists $\delta_x > 0$ such that $d_B(f(x),f(y)) < \epsilon_x$ whenever $d_A(x,y) < \delta_x$. The function is said to be *continuous* if it is continuous at all points $x \in A$.

1.5 Exercises

(a) Let A be a metric space with metric d. Let $y \in A$. Show that the function $f\colon A \to \mathbb{R}$ defined by $f(x) = d(x,y)$ is continuous where $\mathbb{R}$ has the usual metric.

(b) Let M be the metric space $(\mathbb{R},d)$ where d is the usual euclidean metric. Let M_0 be the metric space $(\mathbb{R},d_0)$ where d_0 is the discrete metric, i.e.

$$d_0(x,y) = \begin{cases} 0 \text{ if } x = y, \\ 1 \text{ if } x \neq y. \end{cases}$$

Show that all functions $f\colon M_0 \to M$ are continuous. Show that there does not exist any injective continuous function from M to M_0.

It is often true that by changing the metric on A or B we do not change the set of continuous functions from A to B. For examples see the following exercises.

1.6 Exercises

(a) Let A,B be metric spaces with metrics d and d_B respectively. Let d_r be the metric on A as given in Exercise 1.3(d) (i.e. $d_r(x,y) = rd(x,y)$). Let f be a function from A to B. Prove that f is continuous with respect to the metric d on A if and only if it is continuous with respect to the metric d_r on A.

(b) As (a) but replace d_r by the metric d' of Exercise 1.3(e).

So distance is not the important criterion for whether or not a function is continuous. It turns out that the concept of an 'open set' is what matters.

1.7 Definition

A subset U of a metric space (A,d) is said to be *open* if for all $x \in U$ there exists an $\epsilon_x > 0$ such that if $y \in A$ and $d(y,x) < \epsilon_x$ then $y \in U$.

In other words U is open if for all $x \in U$ there exists an $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) = \{ y \in A; d(y,x) < \epsilon_x \} \subseteq U$.

An example of an open set in $\mathbb{R}$ is $(0,1) = \{ x \in \mathbb{R}; 0 < x < 1 \}$. In $\mathbb{R}^2$

the following are open sets:

$$\{ (x,y) \in \mathbb{R}^2 ; x^2 + y^2 < 1 \} , \ \{ (x,y) \in \mathbb{R}^2 ; x^2 + y^2 > 1 \} ,$$
$$\{ (x,y) \in \mathbb{R}^2 ; 0 < x < 1, 0 < y < 1 \} .$$

1.8 Exercises

(a) Show that $B_\epsilon(x)$ is always an open set for all x and all $\epsilon > 0$.

(b) Which of the following subsets of $\mathbb{R}^2$ (with the usual topology) are open?

$$\{ (x,y); x^2 + y^2 < 1 \} \cup \{ (1,0) \} , \quad \{ (x,y); x^2 + y^2 \leq 1 \} ,$$
$$\{ (x,y); |x| < 1 \} , \quad \{ (x,y); x + y < 0 \} ,$$
$$\{ (x,y); x + y \geq 0 \} , \quad \{ (x,y); x + y = 0 \} .$$

(c) Show that if $\mathscr{F}$ is the family of open sets arising from a metric space then

(i) The empty set $\emptyset$ and the whole set belong to $\mathscr{F}$,

(ii) The intersection of two members of $\mathscr{F}$ belongs to $\mathscr{F}$,

(iii) The union of *any* number of members of $\mathscr{F}$ belongs to $\mathscr{F}$.

(d) Give an example of an infinite collection of open sets of $\mathbb{R}$ (with the usual metric) whose intersection is not open.

Using the concept of an open set we have the following crucial result.

1.9 Theorem

A function $f: M_1 \to M_2$ between two metric spaces is continuous if and only if for all open sets U in M_2 the set $f^{-1}(U)$ is open in M_2.

This result says that f is continuous if and only if *inverse* images of open sets are open. It does *not* say that images of open sets are open.

Proof Let d_1 and d_2 denote the metrics on M_1 and M_2 respectively. Suppose that f is continuous and suppose that U is an open subset of M_2. Let $x \in f^{-1}(U)$ so that $f(x) \in U$. Now, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$ since U is open. The continuity of f assures that there is a $\delta > 0$ such that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon,$$

or in other words $f(B_\delta(x)) \subseteq B_\epsilon(f(x)) \subseteq U$ which means that $B_\delta(x) \subseteq f^{-1}(U)$. Since this is so for all $x \in f^{-1}(U)$ it follows that $f^{-1}(U)$ is an open subset of M_1.

Conversely let $x \in M_1$; then for all $\epsilon > 0$ the set $B_\epsilon(f(x))$ is an open subset of M_2 so that $f^{-1}(B_\epsilon(f(x)))$ is an open subset of M_1. But this means that since $x \in f^{-1}(B_\epsilon(f(x)))$ there is some $\delta > 0$ with $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$, i.e.

$f(B_\delta(x)) \subseteq B_\epsilon(f(x))$. In other words there is a $\delta > 0$ such that $d_2(f(x),f(y)) < \epsilon$ whenever $d_1(x,y) < \delta$, i.e. f is continuous

This theorem tells us, in particular, that if two metrics on a set give rise to the same family of open sets then any function which is continuous using one metric will automatically be continuous using the other. Thus Exercises 1.6 can be rephrased as 'show that the metrics d, d_r and d' give rise to the same family of open sets'.

1.10 Exercise

Which of the metrics $d(x,y) = \Sigma\,|x_i - y_i|$, $d(x,y) = \max |x_i - y_i|$ on $\mathbb{R}^n$ gives rise to the same family of open sets as that arising from the usual metric on $\mathbb{R}^n$?

From the above we see that in order to study continuity between metric spaces it is the family of open sets in each metric space that is important, and not the metric itself. This leads to the following idea: Given a set X choose a family $\mathscr{F}$ of subsets of X and call these the 'open sets' of X. This gives us an object $(X,\mathscr{F})$ consisting of a set X together with a family $\mathscr{F}$ of subsets of X. Continuity between two such objects $(X,\mathscr{F})$, $(Y,\mathscr{F}')$ could then be defined by saying that $f\colon X \to Y$ is continuous if $f^{-1}(U) \in \mathscr{F}$ whenever $U \in \mathscr{F}'$. Naturally if we allowed arbitrary families then we would not get any interesting mathematics. We therefore insist that the family $\mathscr{F}$ of 'open sets' obeys some simple rules: rules that the family $\mathscr{F}$ of open sets arising from a metric space obey (Exercise 1.8(c)). These are

(i) (for convenience) the empty set $\emptyset$ and the whole set belong to $\mathscr{F}$,
(ii) the intersection of two members of $\mathscr{F}$ belongs to $\mathscr{F}$,
(iii) the union of *any* number of members of $\mathscr{F}$ belongs to $\mathscr{F}$.

The 'structure' associated with a set X, referred to at the beginning of this chapter, is simply a family $\mathscr{F}$ of subsets of X satisfying the above three properties. This is the starting point of topology.