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Excerpt
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CHAPTER 0

GUIDE

This short guide is intended to help the reader to find his way about the book.

As we indicated in the Foreword, the book consists of a basic course on affine approximation, which we refer to colloquially as 'the Dieudonné course', linked at various stages with various geometrical examples whose construction is algebraic and to which the topological and differential theorems are applied.

Chapters 1 and 2, on sets, maps and the various number systems serve to fix basic concepts and notations. The Dieudonné course proper starts at Chapter 3. Since the intention is to apply linear algebra in analysis, one has to start by studying linear spaces and linear maps, and this is done here in Chapters 3 to 7 and in the first part of Chapter 8. Next one has to set up the theory of topological spaces and continuous maps. This is done in Chapter 16, this being prefaced, for motivational and for technical reasons, by a short account of normed linear spaces in Chapter 15. The main theorems of linear approximation are then stated and proved in Chapters 18 and 19, paralleling Chapters 8 and 10, respectively, of Prof. Dieudonné's book [14].

The remainder of the book is concerned with the geometry. We risk a brief consideration of the simplest geometrical examples here, leaving the reader to come back and fill in the details when he feels able to do so.

Almost the simplest example of all is the unit circle, S^1, in the plane $\mathbf{R}^2$. This is a smooth curve. It also has a group structure, if one interprets its points as complex numbers of absolute value 1, the group product being multiplication. This group may be identified in an obvious way with the group of rotations of $\mathbf{R}^2$, or indeed of S^1 itself, about the origin.

What about $\mathbf{R}^3$? The situation is now more complicated, but the ingredients are analogous. The complex numbers are replaced, not by a three-dimensional, but by a four-dimensional algebra called the quaternion algebra and identifiable with $\mathbf{R}^4$ just as the complex algebra is identifiable with $\mathbf{R}^2$, and the circle group is replaced by the group of quaternions of absolute value 1, this being identifiable with the unit sphere, S^3, in $\mathbf{R}^4$, the set of points in $\mathbf{R}^4$ at unit distance from 0. As for the group of rotations of $\mathbf{R}^3$, this turns out to be identifiable not with S^3 but with the space obtained by identifying each point of the sphere with the point antipodal to it.

An example of how this model of the group of rotations of $\mathbf{R}^3$ can be used is the following.

Suppose one rotates a solid body continuously about a point. Then the axial rotation required to get from the initial position of the body to the position at any given moment will vary continuously. The initial position may be represented on S^3 by one of the two points representing the identity rotation, say the real quaternion 1 which we may think of as the North pole of S^3. The subsequent motion of the body may then be represented by a continuous path on the sphere. What one can show is that after a rotation through an angle 2π about any axis one arrives at the South pole, the real quaternion -1, After a further full rotation about the same axis one arrives back at 1. There are various vivid illustrations of this, one of the simplest being the soup plate trick, in which the performer rotates a soup plate lying horizontally on his hand through an angle 2π about the vertical line through its centre. His arm is then necessarily somewhat twisted. A further rotation of the plate through 2π about the vertical axis surprisingly brings the arm back to the initial position. The twisting of the arm at any point in time provides a record of a possible path of the plate from its initial to its new position, this being recorded on the sphere by a path from the initial point to the new point. As the arm twists, so the path varies continuously. The reason why it is possible for the arm to return to the initial position after a rotation of the hand through 4π is that it is possible to deform the path of the actual rotation, namely a great circle on S^3, continuously on the sphere to a point. This is not possible in the two-dimensional analogue when the group of rotations is a circle (see Exercise 16.106!). A great circle on S^1 is necessarily S^1 itself, and this is not deformable continuously on itself to a point.

The thing to be noticed here is that the topological (or continuous) features of the model are as essential to its usefulness as the algebraic or geometrical ones.

It is natural to ask, what comes next? For example, which algebras do for higher-dimensional spaces what the complex numbers and the quaternions do for $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively? The answer is provided by the Clifford algebras, and this is the motivation for their study here. Our treatment of quadratic forms and their Clifford algebras in Chapters 9, 10 and 13 is somewhat more general, for we consider there not only the positive-definite quadratic forms necessary for the description of the euclidean (or Pythagorean) distance, but also indefinite forms such as the four-dimensional quadratic form

$$(x,y,z,t) \rightsquigarrow x^2 + y^2 + z^2 - t^2,$$

which arises in the theory of relativity. The Lorentz groups also are introduced.

Chapter 14 contains an alternative answer to the 'what comes next' question.

Analogues of the rotation groups arise in a number of contexts, and Chapter 11 is devoted to their study. One of the principal reasons for the generality given here is to be found towards the end of the chapter on Clifford algebras, Chapter 13. On a first reading it might, in fact, be easier to tackle the early part of Chapter 13 first, before a detailed study of Chapter 11.

Besides the spheres and the rotation groups, there are other examples of considerable interest, such as the projective spaces, their generalization the Grassmannians, and subspaces of them known as the quadrics, defined by quadratic forms. All these are defined and studied in Chapter 8 (the latter part) and Chapter 12.

There remain three chapters to consider, and all are strongly geometrical in flavour. In Chapter 17 important topological features such as the compactness or connectedness or otherwise of the examples introduced earlier are studied, while Chapter 20 is devoted to the study of the smoothness of the same examples. The latter chapter introduces the very important concepts of smooth manifolds and their tangent spaces and of Lie groups and Lie algebras. Chapter 21, on triality, is a new chapter of which we have said something in the Foreword. It leads on naturally from Chapters 13 and 14 and also from Chapter 20 and provides many important examples of transitive group actions and special isomorphisms between groups as well as being an introduction to one of the exceptional Lie groups, G_2.

And now a word to the experts about what is not included. In the algebraic direction we stop short of anything involving eigenvalues, while in the analytical direction the exponential function only turns up in an occasional example. The differential geometry is almost wholly concerned with the first differential and there is nothing whatsoever on integration. Riemannian metrics are nowhere mentioned, nor is there anything on curvature, nor on connections. Finally, in the topological direction there is no discussion of the fundamental group nor of the classification of surfaces. All these topics are, however, more than adequately treated in the existing literature.

CHAPTER 1

MAPS

The language of sets and maps is basic to any presentation of mathematics. Unfortunately, in many elementary school books sets are discussed at length while maps are introduced clumsily, if at all, at a rather late stage in the story. In this chapter, by contrast, maps are introduced as early as possible. Also, by way of a change, more prominence than is usual is given to the von Neumann construction of the set of natural numbers.

Most of the material is standard. Non-standard notations include $f_{\vdash}$ and $f^{\dashv}$, to denote the *forward* and *backward* maps of subsets induced by a map f, and $X!$, to denote the set (and in Chapter 2 the group) of permutations of a set X. The notation ω for the set of natural numbers is that used in [21] and in [34]. An alternative notation in common use is $\mathbf{N}$.

Membership

Membership of a *set* is denoted by the symbol $\in$, to be read as an abbreviation for 'belongs to' or 'belonging to' according to its grammatical context. The phrase 'x is a member of X' is denoted by $x \in X$. The phrase 'x is not a member of X' is denoted by $x \notin X$. A member of a set is also said to be an *element* or a *point* of the set. Sets X and Y are *equal*, $X = Y$, if, and only if, each element of X is an element of Y and each element of Y is an element of X. Otherwise the sets are *unequal*, $X \neq Y$. Sets X and Y *intersect* or *overlap* if they have a common member and are *mutually disjoint* if they have no common member.

A set may have no members. It follows at once from the definition of equality for sets that there is only one such set. It is called the *null* or *empty set* or the *number zero* and is denoted by $\emptyset$, or by 0, though the latter symbol, having many other uses, is best avoided when we wish to think of the null set as a set, rather than as a number.

An element of a set may itself be a set. It is, however, not logically permissible to speak of the set of all sets. See Exercise 1.60 (the Russell Paradox).

Sometimes it is possible to list all the members of a set. In such a case the set may be denoted by the list of its members inside $\{\ \}$, the order in which the elements are listed being irrelevant. For example, $\{x\}$ denotes the set whose sole member is the element x, while $\{x,y\}$ denotes the set whose sole members are the elements x and y. Note that $\{y,x\} = \{x,y\}$ and that $\{x,x\} = \{x\}$. The set $\{x\}$ is not the same thing as the element x, though one is often tempted to ignore the distinction for the sake of having simpler notations. For example, let $x = 0$ $(= \emptyset)$. Then $\{0\} \neq 0$, for $\{0\}$ has a member, namely 0, while 0 has no members at all. The set $\{0\}$ will be denoted by 1 and called the *number one* and the set $\{0,1\}$ will be denoted by 2 and called the *number two*.

Maps

Let X and Y be sets. A *map* $f: X \longrightarrow Y$ associates to each element $x \in X$ a unique element $f(x) \in Y$.

Suppose, for example, that X is a class of students and that Y is the set of desks in the classroom. Then any seating arrangement of the members of X at the desks of Y may be regarded as a map of X to Y (though not as a map of Y to X): to each student there is associated the desk he or she is sitting at. We shall refer to this briefly as a *classroom map*.

Maps $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ are said to be *equal* if, and only if, $X' = X$, $Y' = Y$ and, for each $x \in X$, $f'(x) = f(x)$. The sets X and Y are called, respectively, the *domain* and the *target* of the map f. For any $x \in X$, the element $f(x)$ is said to be the *value* of f at x or the *image* of x by f, and we say informally that f *sends* x to $f(x)$. We denote this by $f; x \rightsquigarrow f(x)$ or, if the domain and target of f need mention, by $f: X \longrightarrow Y;$ $x \rightsquigarrow f(x)$.

The arrow $\mapsto$ is used by many authors in place of $\rightsquigarrow$. The arrow $\longrightarrow$ is also used, but this can lead to confusion when one is discussing maps between sets of sets. For our use of the arrow $\rightarrowtail$, and the term *source* of a map, see page 39. The *image* of a map is defined below, on page 8. The word 'range' has not been used here, either to denote the target or the image of a map. This is because both usages are current. By avoiding the word we avoid confusion.

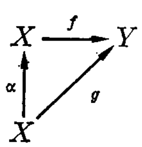
To any map $f: X \longrightarrow Y$ there is associated an equation $f(x) = y$. The map f is said to be *surjective* or *a surjection* if, for each $y \in Y$, there is some $x \in X$ such that $f(x) = y$. It is said to be *injective* or *an injection*, if, for each $y \in Y$, there is at most one element $x \in X$, though possibly none, such that $f(x) = y$. The map fails to be surjective if there exists an element $y \in Y$ such that the equation $f(x) = y$ has no *solution* $x \in X$, and fails to be injective if there exist distinct $x, x' \in X$ such that $f(x') = f(x)$. For example, a classroom map fails to be surjective if there is an empty desk and fails to be injective if there is a desk at which there is more than one student. The map f is said to be *constant* if, for all $x, x' \in X$, $f(x') = f(x)$.

If the map $f: X \longrightarrow Y$ is both *surjective* and *injective*, it is said to be *bijective* or to be *a bijection*. In this case the equation $f(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. In the classroom *each* desk is occupied by *one* student only.

An injection, or more particularly a bijection, $f: X \longrightarrow Y$ may be thought of as a labelling device, each element $x \in X$ being labelled by its image $f(x) \in Y$. In an injective seating arrangement each student may, without ambiguity, be referred to by the desk he occupies.

A map $f: X \longrightarrow X$ is said to be a *transformation* of X, and a bijective map $\alpha: X \longrightarrow X$ a *permutation* of X.

Example 1.1. Suppose that $f: X \longrightarrow Y$ is a bijection. Then a second bijection $g: X \longrightarrow Y$ may be introduced in one of three ways; directly, by stating $g(x)$ for each $x \in X$, or indirectly, either in terms of a permutation $\alpha: X \longrightarrow X$, with $g(x)$ defined to be $f(\alpha(x))$, or in terms of a permutation $\beta: Y \longrightarrow Y$, with $g(x)$ defined to be $\beta(f(x))$. These last two possibilities are illustrated by the diagrams



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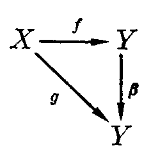

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(The maps f and g may be thought of as bijective classroom maps on successive days of the week. The problem is, how to tell the class on Monday the seating arrangement preferred on Tuesday. The example indicates three ways of doing this. For the proof that g defined in either of the last two ways is bijective, see Cor. 1.4 or Prop. 1.6 below.) □

Example 1.1 illustrates the following fundamental concept.

Let $f: X \longrightarrow Y$ and $g: W \longrightarrow X$ be maps. Then the map $W \longrightarrow Y$; $w \rightsquigarrow f(g(w))$ is called the *composite* fg (read '*f following g*') of f and g. (An alternative notation for fg is $f \circ g$. See also page 30.) We need not restrict ourselves to two maps. If, for example, there is also a map $h: V \longrightarrow W$, then the map $V \longrightarrow Y$; $v \rightsquigarrow f(g(h(v)))$ will be called the composite fgh of f, g and h.

Prop. 1.2. For any maps $f: X \longrightarrow Y$, $g: W \longrightarrow X$ and $h: V \longrightarrow W$

$$f(gh) = fgh = (fg)h.$$

Proof For all $v \in V$,

$$(f(gh))(v) = f((gh)(v)) = f(g(h(v)))$$

and

$$((fg)h)(v) = (fg)(h(v)) = f(g(h(v))). \qquad \square$$

Prop. 1.3. Let $f: X \longrightarrow Y$ and $g: W \longrightarrow X$ be maps. Then

(i) f and g surjective $\Rightarrow$ fg surjective
(ii) f and g injective $\Rightarrow$ fg injective
(iii) fg surjective $\Rightarrow$ f surjective
(iv) fg injective $\Rightarrow$ g injective.

We prove (ii) and (iii), leaving (i) and (iv) as exercises.

Proof of (ii) Let f and g be injective and suppose that $a, b \in W$ are such that $fg(a) = fg(b)$. Since f is injective, $g(a) = g(b)$ and, since g is injective, $a = b$. So, for all $a, b \in W$,

$$fg(a) = fg(b) \Rightarrow a = b.$$

That is, fg is injective.

Proof of (iii) Let fg be surjective and let $y \in Y$. Then there exists $w \in W$ such that $fg(w) = y$. So $y = f(x)$, where $x = g(w)$. So, for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$. That is, f is surjective. $\square$

Cor. 1.4. Let $f: X \longrightarrow Y$ and $g: W \longrightarrow X$ be maps. Then

(i) f and g bijective $\Rightarrow$ fg bijective
(ii) fg bijective $\Rightarrow$ f surjective and g injective. $\square$

Bijections may be handled directly. The bijection $1_X: X \longrightarrow X$; $x \rightsquigarrow x$ is called the *identity map* or *identity permutation* of the set X and a map $f: X \longrightarrow Y$ is said to be *invertible* if there exists a map $g: Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$.

Prop. 1.5. A map $f: X \longrightarrow Y$ is invertible if, and only if, it is bijective.

(There are two parts to the proof, corresponding to 'if' and 'only if', respectively.) $\square$

Note that $1_Y f = f = f\,1_X$, for any map $f: X \longrightarrow Y$. Note also that if $g: Y \longrightarrow X$ is a map such that $gf = 1_X$ and $fg = 1_Y$, then it is the only one, for if g' is another such then $g' = g'1_Y = g'fg = 1_X g = g$. When such a map exists it is called the *inverse* of f and denoted by f^{-1}.

Prop. 1.6. Let $f: X \longrightarrow Y$ and $g: W \longrightarrow X$ be invertible. Then f^{-1}, g^{-1} and fg are invertible, $(f^{-1})^{-1} = f$, $(g^{-1})^{-1} = g$ and $(fg)^{-1} = g^{-1}f^{-1}$. $\square$

Let us return for a moment to Example 1.1. The bijections f and g in that example were related by permutations $\alpha : X \rightarrow X$ and $\beta : Y \rightarrow Y$ such that $g = f\alpha = \beta f$. It is now clear that in this case α and β exist and are uniquely determined by f and g. In fact $\alpha = f^{-1}g$ and $\beta = g f^{-1}$.

Note in passing that if $h : X \rightarrow Y$ is a third bijection, then $h f^{-1} = (h g^{-1})(g f^{-1})$, while $f^{-1} h = (f^{-1} g)(g^{-1} h)$, the order in which the permutations are composed in the one case being the reverse of the order in which they are composed in the other case.

Subsets and quotients

If each element of a set W is also an element of a set X, then W is said to be a *subset* of X, this being denoted either by $W \subset X$ or by $X \supset W$. For example, $\{1,2\} \subset \{0,1,2\}$. The injective map $W \rightarrow X;\ w \rightsquigarrow w$ is called the *inclusion* of W in X.

In practice a subset is often defined in terms of the truth of some proposition. The subset of a set X consisting of all x in X for which some proposition $P(x)$ concerning x is true is normally denoted by $\{x \in X : P(x)\}$, though various abbreviations are in use in special cases. For example, a map $f : X \rightarrow Y$ defines various subsets of X and Y. The set $\{y \in Y : y = f(x)$, for some $x \in X\}$, also denoted more briefly by $\{f(x) \in Y : x \in X\}$, is a subset of Y called the *image* of f and denoted by $\operatorname{im} f$. It is non-null if X is non-null. It is also useful to have a short name for the set $\{x \in X : f(x) = y\}$, where y is an element of Y. This will be called the *fibre* of f over y. It is a subset of X, possibly null if f is not surjective. The fibres of a map f are sometimes called the *levels* or *contours* of f, especially when the target of f is $\mathbf{R}$, the real numbers (introduced formally in Chapter 2). The set of non-null fibres of f is called the *coimage* of f and is denoted by $\operatorname{coim} f$.

A subset of a set X that is neither null nor the whole of X is said to be a *proper subset* of X.

The elements of a set may themselves be sets or maps. In particular it is quite possible that two elements of a set may themselves be sets which intersect. This is the case with the *set of all subsets* of a set X, Sub X (known also as the *power set* of X for reasons given on pages 10 (Prop 1.11) and 21).

Consider, for example,

$$\text{Sub}\,\{0,1,2\} = \{0,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\},$$

where $0 = \emptyset$, $1 = \{0\}$ and $2 = \{0,1\}$ as before. The elements $\{0,1\}$ and $\{0,2\}$ are subsets of $\{0,1,2\}$, which intersect each other. A curious fact about this example is that each *element* of $\{0,1,2\}$ is also a *subset* of $\{0,1,2\}$ (though the converse is not, of course, true). For example, $2 = \{0,1\}$ is both an element of $\{0,1,2\}$ and a subset of $\{0,1,2\}$. We shall return to this later in Prop. 1.34.

Frequently one classifies a set by dividing it into mutually disjoint subsets. A map $f: X \longrightarrow Y$ will be said to be a *partition* of the set X, and Y to be the *quotient* of X by f, if f is surjective, if each element of Y is a subset of X, and if the fibre of f over any $y \in Y$ is the set y itself.

For example, the map $\{0,1,2\} \longrightarrow \{\{0,1\},\{2\}\}$ sending 0 and 1 to $\{0,1\}$ and 2 to $\{2\}$ is a partition of $\{0,1,2\}$ with quotient the set $\{\{0,1\},\{2\}\}$ of subsets of $\{0,1,2\}$.

The following properties of partitions and quotients are easily proved.

Prop. 1.7. Let X be a set. A subset $\mathscr{S}$ of Sub X is a quotient of X if, and only if, the null set is not a member of $\mathscr{S}$, each $x \in X$ belongs to some $A \in \mathscr{S}$ and no $x \in X$ belongs to more than one $A \in \mathscr{S}$. □

Prop. 1.8. A partition f of a set X is uniquely determined by the quotient of X by f. □

Any map $f: X \longrightarrow Y$ with domain a given set X induces a partition of X, as follows.

Prop. 1.9. Let $f: X \longrightarrow Y$ be a map. Then coim f, the set of non-null fibres of f, is a quotient of X.

Proof By definition the null set is not a member of coim f. Also, each $x \in X$ belongs to the fibre of f over $f(x)$. Finally, x belongs to no other fibre, since the statement that x belongs to the fibre of f over y implies that $y = f(x)$. □

It is occasionally convenient to have short notations for the various injections and surjections induced by a map $f: X \longrightarrow Y$. Those we shall use are the following:

f_{inc} for the inclusion of im f in Y,
f_{par} for the partition of X on to coim f,
f_{sur} for f 'made surjective', namely the map

$$X \longrightarrow \operatorname{im} f;\ x \rightsquigarrow f(x),$$

f_{inj} for f 'made injective', namely the map

$$\operatorname{coim} f \longrightarrow Y;\ f_{\text{par}}(x) \rightsquigarrow f(x),$$

and, finally,

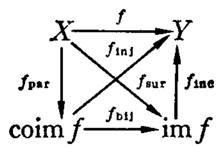
f_{bij} for f 'made bijective', namely the map

$$\operatorname{coim} f \longrightarrow \operatorname{im} f;\ f_{\text{par}}(x) \rightsquigarrow f(x).$$

Clearly,

$$f = f_{\text{inc}} f_{\text{sur}} = f_{\text{inj}} f_{\text{par}} = f_{\text{inc}} f_{\text{bij}} f_{\text{par}}.$$

These maps may be arranged in the diagram



Such a diagram, in which any two routes from one set to another represent the same map, is said to be a *commutative diagram*. For example, the triangular diagrams on page 6 are commutative. For the more usual use of the word 'commutative' see page 15.

The composite $fi: W \to Y$ of a map $f: X \to Y$ and an inclusion $i: W \to X$ is said to be the *restriction* of f to the subset W of X and denoted also by $f \mid W$. The target remains unaltered.

If maps $f: X \to Y$ and $g: Y \to X$ are such that $fg = 1_Y$ then, by Prop. 1.3, f is surjective and g is injective. The injection g is said to be a *section* of the surjection f. It selects for each $y \in Y$ a *single* $x \in X$ such that $f(x) = y$.

It is assumed that any surjection $f: X \to Y$ has a section $g: Y \to X$, this assumption being known as the *axiom of choice*.

Notice that to prove that a map $g: Y \to X$ is the inverse of a map $f: X \to Y$ it is not enough to prove that $fg = 1_Y$. One also has to prove that $gf = 1_X$.

Prop. 1.10. Let $g, g': Y \to X$ be sections of $f: X \to Y$. Then $g = g' \Leftrightarrow \operatorname{im} g = \operatorname{im} g'$.

Proof $\Rightarrow$: Clear.

$\Leftarrow$: Let $y \in Y$. Since $\operatorname{im} g = \operatorname{im} g'$, $g(y) = g'(y')$ for some $y' \in Y$. But $g(y) = g'(y') \Rightarrow fg(y) = fg'(y') \Rightarrow y = y'$. That is, for all $y \in Y$, $g(y) = g'(y)$. So $g = g'$. □

A map $g: B \to X$ is said to be a *section* of the map $f: X \to Y$ *over the subset* B of Y if, and only if, $fg: B \to Y$ is the inclusion of B in Y.

The set of maps $f: X \to Y$, with given sets X and Y as domain and target respectively, is sometimes denoted by Y^X, for a reason which will be given on page 21.

Prop. 1.11. Let X be any set. Then the map

$$2^X \to \operatorname{Sub} X : f \rightsquigarrow \{x \in X : f(x) = 0\}$$

is bijective. □