Cambridge University Press
978-0-521-29746-2 - The Theory of Partial Differential Equations
Siferu Mizohata
Excerpt
More information

CHAPTER 1

Fourier series and Fourier transforms

1 *Fourier series*

Let $f(x)$ be a continuous function, with period 2π, which we try to represent as follows:

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1.1}$$

Let us assume that this right hand series is uniformly convergent. As is well known, we can integrate this term by term to obtain coefficients a_0, a_n and b_n. First we note that

$$\int_0^{2\pi}\cos mx\cos nx\,\mathrm{d}x=\tfrac{1}{2}\int_0^{2\pi}(\cos(m-n)x+\cos(m+n)x)\,\mathrm{d}x$$

$$=\begin{cases}0 & (m\neq n),\\ \pi & (m=n).\end{cases}$$

$$\int_0^{2\pi}\cos mx\sin nx\,\mathrm{d}x=\tfrac{1}{2}\int_0^{2\pi}(\sin(m+n)x-\sin(m-n)x)\,\mathrm{d}x=0.$$

$$\int_0^{2\pi}\sin mx\sin nx\,\mathrm{d}x=\tfrac{1}{2}\int_0^{2\pi}(\cos(m-n)x-\cos(m+n)x)\,\mathrm{d}x$$

$$=\begin{cases}0 & (m\neq n),\\ \pi & (m=n).\end{cases}$$

$$\int_0^{2\pi}\cos mx\,\mathrm{d}x=\int_0^{2\pi}\sin mx\,\mathrm{d}x=0\quad(m\neq 0).$$

In other words, the series of functions 1, $\cos x$ $\sin x$, $\cos 2x$, $\sin 2x, \ldots$ is an orthogonal system. Therefore, multiplying both sides of (1.1) by 1, $\cos nx$ and $\sin nx$ in turn, and integrating, we obtain

$$\left.\begin{aligned} \pi a_0 &= \int_0^{2\pi} f(x)\,dx\,, \\ \pi a_n &= \int_0^{2\pi} f(x)\cos nx\,dx\,, \\ \pi b_n &= \int_0^{2\pi} f(x)\sin nx\,dx\,, \end{aligned}\right\} \tag{1.2}$$

and thus we can calculate a_0, a_n, and b_n.

This poses the following problem. Suppose that $f(x)$ is a continuous function, and that we have determined a_0, a_n, b_n (the Fourier coefficients) by the use of (1.2). Then let

$$\varphi(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,. \tag{1.3}$$

Is $\varphi(x)$ always meaningful and identical to the original $f(x)$, i.e. can we write $f(x) = \varphi(x)$? Generally speaking, the answer is in the negative. However, given $f(x)$, we can at least calculate the Fourier coefficients, and write

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

We call this the *Fourier series* of $f(x)$.

Historically, the first mathematician to study the series rigorously was Dirichlet. Later, Riemann and Lebesgue laid the foundations of the theory. According to Lebesgue, given $f(x)$, the Fourier series can be defined when $f(x)$ is Lebesgue-integrable. In the following arguments, unless stated to the contrary, we consider $f(x)$ to be Lebesgue-integrable. We say simply that $f(x)$ is *summable*.

We now prove the following theorem.

THEOREM 1.1 *(Lebesgue)*. If $f_1(x)$ and $f_2(x)$ have the same Fourier coefficients, then $f_1(x) = f_2(x)$ except possibly on a set having measure 0.

Proof (By contradiction). When $f(x) = f_1(x) - f_2(x)$, the Fourier coefficients of $f(x)$ are all zero. Also, we assume $f(x)$ to be continuous and $f(x) \not\equiv 0$. As $f(x)$ is orthogonal to 1, $\cos nx$, and $\sin nx$ $(n = 1, 2, \ldots)$, so it is orthogonal to their linear combinations, which are trigonometric polynomials. So that

$$\int_0^{2\pi} f(x)\,\varphi(x)\,\mathrm{d}x=0,$$

where $\varphi(x)$ is an arbitrary trigonometric polynomial.

Hence we may put $f(x)\geqslant m\,(m>0)$ on $[a, b]$ which lies within $[0, 2\pi]$. (We can, if necessary, replace $f(x)$ with $-f(x)$). $\varphi(x)$ is calculated from the following relations:

$$\varphi=\psi^n,\qquad \psi(x)=1+\cos(x-\tfrac{1}{2}(a+b))-\cos\tfrac{1}{2}(a-b).$$

We note that φ is a trigonometric polynomial and therefore

$$0=\int_0^{2\pi} f\cdot\varphi\,\mathrm{d}x=\int_a^b f\cdot\varphi\,\mathrm{d}x+\int_{[0,\,2\pi]\setminus[a,\,b]} f\cdot\varphi\,\mathrm{d}x,^{\dagger}$$

where the second part of the integration does not exceed $\max|f(x)|(2\pi-(b-a))$ in its absolute value. In fact, in $[0, 2\pi]\setminus[a, b]$ we observe that $|\psi|\leqslant 1$, and it follows that $|\varphi|=|\psi|^n\leqslant 1$. On the other hand, the first part of the integration converges to $+\infty$ as n increases infinitely, the reason being that $f(x)\geqslant m$ on $[a, b]$, and $\psi\geqslant 1+\delta$ on $[a', b']$ where $[a', b']\subset[a, b]$ and $\delta>0$. Hence

$$\int_a^b f\cdot\varphi\,\mathrm{d}x\geqslant\int_{a'}^{b'} f\cdot\varphi\,\mathrm{d}x\geqslant m(1+\delta)^n\int_{a'}^{b'}\mathrm{d}x=m(1+\delta)^n\,(b'-a').$$

This is a contradiction and hence we conclude $f(x)\equiv 0$.

Next, let $f(x)$ be summable. Write

$$F(x)=\int_0^x f(t)\,\mathrm{d}t.$$

By our hypothesis ($f(x)$ is orthogonal to 1) we have $F(2\pi)=0$. We note the following relations:

$$0=\int_0^{2\pi} f(x)\cos mx\,\mathrm{d}x=[F(x)\cos mx]_0^{2\pi}+m\int_0^{2\pi} F(x)\sin mx\,\mathrm{d}x$$
$$=m\int_0^{2\pi} F(x)\sin mx\,\mathrm{d}x,$$

† Translator's note: $[0, 2\pi]-[a, b]$ is the original expression, but the current Russian notation $\setminus$ is more widely used and less confusing.

Cambridge University Press
978-0-521-29746-2 - The Theory of Partial Differential Equations
Siferu Mizohata
Excerpt
More information

$$0=\int_0^{2\pi} f(x)\sin mx\,dx=[F(x)\sin mx]_0^{2\pi}-m\int_0^{2\pi} F(x)\cos mx\,dx$$
$$=-m\int_0^{2\pi} F(x)\cos mx\,dx.$$

Therefore, the Fourier coefficients of $F(x)-C$ are all zero for some constant C. Using facts already proved, we have $F(x)-C\equiv 0$. Since $F(0)=0$, $F(x)\equiv 0$.

Now, recall the well-known theorem of Lebesgue $F'(x)=f(x)$ is true 'almost everywhere.' We finally† conclude that $f(x)=0$ 'almost everywhere' on $[0, 2\pi]$.

Q.E.D.

Note. The fact shows that the system of trigonometric functions $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots\}$ is complete on $L^1[0, 2\pi]$.

2. *Dirichlet's Integrals* ‡

First we state some important facts.

Abelian transform

Let $\{u_0, u_1, \ldots, u_n\}$, $\{v_0, v_1, \ldots, v_n\}$ be two number sequences where $u_0\geqslant u_1 \geqslant u_2\geqslant\cdots\geqslant u_n\geqslant 0$. If $\sigma_p=v_0+v_1+\cdots+v_p$ $(p=0, 1, 2, \ldots n)$, then
$$(\min_{p=0,\ldots,n} \sigma_p)\, u_0\leqslant S\equiv u_0v_0+u_1v_1+\cdots+u_nv_n\leqslant (\max_{p=0,\ldots,n} \sigma_p)\, u_0.$$

Proof.

$$S=u_0v_0+\cdots+u_nv_n=u_0\sigma_0+u_1(\sigma_1-\sigma_0)+u_2(\sigma_2-\sigma_1)+\cdots+u_n(\sigma_n-\sigma_{n-1})$$
$$=\sigma_0(u_0-u_1)+\sigma_1(u_1-u_2)+\cdots+\sigma_{n-1}(u_{n-1}-u_n)+\sigma_nu_n,$$

† Translator's note: The theorem quoted here is: 'If $f(x)$ is any integrable function, its indefinite integral $F(x)$ has, almost everywhere, a finite differential coefficient equal to $f(x)$.' [See Titchmarsh: *Theory of Functions*. Oxford Univ. Press.]

‡ Translator's note: Dirichlet integral is *not* explicitly defined in the original context in this section. On pages 9 and 20 we probably have to note that

$$S_n=\frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(\theta)\frac{\sin(2n+1)\frac{1}{2}(x-\theta)}{2\sin\frac{1}{2}(x-\theta)}d\theta$$

is the *Dirichlet integral.*

and

$$u_0-u_1\geqslant 0, \quad u_1-u_2\geqslant 0, \cdots, u_n\geqslant 0.$$

It follows that

$$\min(\sigma_0, \sigma_1, \cdots, \sigma_n)\, u_0\leqslant S\leqslant \max(\sigma_0, \sigma_1, \cdots, \sigma_n)\, u_0.$$

Q.E.D.

Changing this to integral form, we obtain the following theorem.

THEOREM 1.2 *(Bonnet)*. Suppose $f(x)$ is Riemann-integrable on a finite interval $[a, b]$ and $\varphi(x)$ is a positive non-increasing function. Under these conditions, there exists a $\xi\in[a, b]$ such that

$$\int_a^b \varphi(x)\, f(x)\, dx=\varphi(a+0)\int_a^\xi f(x)\, dx.$$

Proof. If $\varphi(x)$ is discontinuous at $x=a$, we may replace the value at $x=a$, setting $\varphi(a)=\varphi(a+0)$ without changing the value of the integration which appears on the left hand side of the above equation.

We can, therefore, assume that $\varphi(a)=\varphi(a+0)$ at the outset. We partition $[a, b]$ into n equal intervals

$$a=x_0<x_1<x_2<\cdots<x_n=b, \quad h=x_i-x_{i-1}=(b-a)/n.$$

By the Abelian transform we have

$$\varphi(a+0)\, h\sum_{i=0}^{k'} f(x_i)\leqslant h\sum_{i=0}^{n-1}\varphi(x_i)\, f(x_i)\leqslant \varphi(a+0)\, h\sum_{i=1}^{k} f(x_i).$$

For an arbitrary $\varepsilon\,(>0)$, if h is sufficiently small, we establish

$$h\sum_{i=0}^{k'} f(x_i)\leqslant \max_\xi \int_a^\xi f(x)\, dx+\varepsilon.$$

In fact, from the definition of integration we have

$$\left|h\sum_{i=0}^{k'} f(x_i)-\int_a^{a+(k'+1)h} f(x)\, dx\right|\leqslant\varepsilon,$$

which is always true irrespective of the value of k' provided that h is sufficiently small. Therefore, for $h\to 0$ we have

$$\varphi(a+0)\,[\min_\xi\int_a^\xi f(x)\, dx-\varepsilon]\leqslant\int_a^b\varphi(x)\, f(x)\, dx\leqslant\varphi(a+0)\,[\max_\xi\int_a^\xi f(x)\, dx+\varepsilon].$$

Now we know that $\varepsilon(>0)$ is arbitrary and that $\int_a^\xi f(x)\,\mathrm{d}x$ is in fact a continuous function of ξ.

Q.E.D.

The following theorem is important for various applications.

THEOREM 1.3 *(Riemann–Lebesgue)*. Let $\psi(t)$ be summable on $[a, b]$. Then we have

$$\int_a^b \psi(t)\sin nt\,\mathrm{d}t \to 0 \quad \text{as} \quad n\to\infty .$$

Note. The theorem can be generalized as follows:

$$\int_{-\infty}^{+\infty} \psi(t)\sin \lambda t\,\mathrm{d}t \to 0 \quad \text{as} \quad \lambda\to+\infty ,$$

where λ is a real parameter, and $\psi\in L^1(-\infty, +\infty)$. This is also true when we replace $\sin\lambda t$ by $\cos\lambda t$.

Proof. We partition $[a, b]$ equally by taking endpoints $k\pi/n(k=0, \pm 1, \pm 2, \ldots)$. Then, we separate the integral into several parts

$$\sum_p \left(\int_{2p\pi/n}^{(2p+1)\pi/n} + \int_{(2p+1)\pi/n}^{(2p+2)\pi/n} \right) + \int_a^{a+\delta} + \int_{b-\delta'}^{b} .$$

Here $a+\delta$ is the first endpoint in the form of $2p/n$ lying to the right relative to a, and $b-\delta'$ is the last endpoint in the same form lying to the left relative to b. As $\delta, \delta' < 2\pi/n$, so $\delta, \delta'\to 0$ as $n\to 0$. Also, as $\psi(t)$ is summable, the two final integrals converge to 0 as $n\to 0$. On the other hand, if we put $t-(\pi/n)=t'$, we have

$$\sin nt = \sin n(t'+(\pi/n)) = -\sin nt' .$$

This means that the sum of the first and second integrals can be written as

$$\sum_p \int_{2p\pi/n}^{(2p+1)\pi/n} [\psi(t)-\psi(t+(\pi/n))]\sin nt\,\mathrm{d}t .$$

Moreover, the absolute value of this integral does not exceed

$$\int_a^b |\psi(t)-\psi(t+(\pi/n))|\,\mathrm{d}t .$$

By Lebesgue's lemma, if $\psi(t)$ is summable

$$\int_a^b |\psi(t+h)-\psi(t)|\,dt \to 0 \quad \text{as} \quad h\to 0.$$

It follows that the value of the integral in question converges to 0 as $n\to\infty$.

Q.E.D.

We have used Lebesgue's lemma, a result we shall use later.

We now give a proof.

LEMMA 1.1 *(Lebesgue).* Let $\psi \in L^p(a, b)$ where $p \geqslant 1$, and (a, b) is a finite or infinite interval. We have

$$\int_a^b |\psi(t+h)-\psi(t)|^p\,dt \to 0 \quad \text{as} \quad h\to 0.$$

Proof. It is sufficient to assume that (a, b) is finite, and $\psi(t)$ is bounded. We proceed under these assumptions. First, as $\psi(t)$ is measurable it can be represented as a measurable limit of a sequence of step functions.† Therefore, for an arbitrary $\varepsilon(>0)$ there exists a step function $\varphi(t)$ and

$$\int_a^b |\psi(t)-\varphi(t)|^p\,dt < \varepsilon.$$

Note that for $\varphi(t)$ the properties can be easily established. In fact, for an appropriate partition $a<x_1<x_2<\cdots<x_n=b$, $\varphi(t)$ is constant on each (x_i, x_{i+1}). Therefore, from

$$3^{1-p}\int_a^b |\psi(t+h)-\psi(t)|^p\,dt \leqslant \int_a^b |\varphi(t+h)-\varphi(t)|^p\,dt + \int_a^b |\varphi(t+h)-\psi(t+h)|^p\,dt + \int_a^b |\varphi(t)-\psi(t)|^p\,dt$$

as $h\to 0$, the first term of the right hand side of the inequality tends to 0, and the second and the third are smaller than ε. ε is arbitrary (>0). Hence, the left hand term tends to 0 as $h\to 0$.

Q.E.D.

† Moreover, we can assume that the sequence is uniformly bounded.

Returning to the initial argument, we now consider the n-term sum of the Fourier series of $f(x)$

$$S_n = \tfrac{1}{2}a_0 + \sum_{p=1}^{n} (a_p \cos px + b_p \sin px). \tag{1.4}$$

We give a proof of the fact $S_n \to f(x)$. To this end we note that

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \tfrac{1}{2}\pi. \tag{1.5}$$

Care must be taken in dealing with the equality. We can readily see that as

$$\int_0^{+\infty} \frac{|\sin x|}{x}\, dx = +\infty,$$

$|\sin x|/x$ is *not* summable in $(0, \infty)$, the integral is a so-called 'improper integral'.

In this sense we should write (1.5) as

$$\int_0^{\to +\infty} \frac{\sin x}{x}\, dx = \lim_{A \to +\infty} \int_0^A \frac{\sin x}{x}\, dx = \tfrac{1}{2}\pi. \tag{1.6}$$

We must verify the existence of the limit of (1.5); for this we use Bonnet's theorem (theorem 1.2). In fact,

$$\left|\int_A^{A'} \frac{\sin x}{x}\, dx\right| = \left|\frac{1}{A}\right| \left|\int_A^{\xi} \sin x\, dx\right| \leqslant \left|\frac{2}{A}\right| \to 0 \quad (A \to +\infty). \tag{1.7}$$

In order to conclude that the limit is in fact $\tfrac{1}{2}\pi$, we multiply by a converging factor $e^{-\alpha x}$ $(\alpha > 0)$ to get

$$F(\alpha) = \int_0^{\infty} e^{-\alpha x} \frac{\sin x}{x}\, dx \quad (\alpha > 0).$$

It is obvious that

$$\int_0^{\to +\infty} \frac{\sin x}{x}\, dx = \lim_{\alpha \to +0} F(\alpha),$$

where $F(\alpha)$ is continuously differentiable in $\alpha > 0$, and

$$F'(\alpha) = -\int_0^{\infty} e^{-\alpha x} \sin x\, dx.$$

The right hand term is $-1/(\alpha^2+1)$, therefore, $F(\alpha)=C-\arctan\alpha$ and $F(\alpha)\to 0\,(\alpha\to+\infty)$. Hence, $C=\frac{1}{2}\pi$ and $\lim_{\alpha\to+0} F(\alpha)=\frac{1}{2}\pi$.

Q.E.D.

We are now ready to prove the following well-known theorem.

THEOREM 1.4 *(Jordan–Lebesgue)*. Let $f(x)$ be a function which has period 2π, and is summable in $[0, 2\pi]$. We further assume that $f(x)$ is of bounded variation in the neighbourhood of x. Then we have

$$S_n(x)\to\tfrac{1}{2}[f(x+0)+f(x-0)] \quad (n\to\infty).$$

Moreover, if $f(x)$ is continuous in $[\alpha, \beta]$ and of bounded variation, then $S_n(x)$ converges uniformly to $f(x)$ in an arbitrary interval $[\alpha', \beta']$ which is entirely contained within $[\alpha, \beta]$.

Proof.

$$\begin{aligned} S_n &= \frac{1}{2\pi}\int_\alpha^{\alpha+2\pi} f(\theta)\,d\theta+\frac{1}{\pi}\sum_{p=1}^{n}\left[\cos px\int_\alpha^{\alpha+2\pi} f(\theta)\cos p\theta\,d\theta\right. \\ &\qquad\left.+\sin px\int_\alpha^{\alpha+2\pi} f(\theta)\sin p\theta\,d\theta\right] \\ &= \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(\theta)\left[\tfrac{1}{2}+\sum_{p=1}^{n}\cos p(x-\theta)\right]d\theta \\ &= \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(\theta)\,\frac{\sin(2n+1)\frac{1}{2}(x-\theta)}{2\sin\frac{1}{2}(x-\theta)}\,d\theta.^{\dagger} \end{aligned}$$

Here we have used the well-known formula

$$\tfrac{1}{2}+\sum_{p=1}^{n}\cos 2pt=\frac{\sin(2n+1)t}{2\sin t}. \tag{1.8}$$

In the foregoing expression of S_n, we put $\theta=x+2t$ and change the integral variable from θ to t, then we have

$$S_n=\frac{1}{\pi}\int_\beta^{\beta+\pi} f(x+2t)\,\frac{\sin(2n+1)t}{\sin t}\,dt.$$

† Translator's note: The last integral is the *Dirichlet integral*. We note this here because otherwise the section title has no significance.

α is arbitrary, therefore β is also arbitrary. We put $\beta = -\frac{1}{2}\pi$, separate the integral to $\int_{-\frac{1}{2}\pi}^{0} + \int_{0}^{\frac{1}{2}\pi}$, and change t into $-t$ in the first integral. We now have

$$S_n = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)t}{\sin t}[f(x+2t)+f(x-2t)]\,\mathrm{d}t. \tag{1.9}$$

From (1.8) we note that

$$f(x) = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)t}{\sin t}\cdot 2f(x)\,\mathrm{d}t,$$

therefore (1.9) can be written as

$$\begin{aligned}\pi[S_n - f(x)] &= \int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)t}{\sin t}[f(x+2t)+f(x-2t)-2f(x)]\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)t}{t}\varphi(t)\,\mathrm{d}t, \end{aligned} \tag{1.10}$$

where

$$\varphi(t) = \frac{t}{\sin t}[f(x+2t)+f(x-2t)-2f(x)]. \tag{1.11}$$

Then we make $\delta > 0$ small and write the right hand integral of (1.10) as

$$\pi[S_n - f(x)] = \int_0^{\delta} \frac{\sin(2n+1)t}{t}\varphi(t)\,\mathrm{d}t + \int_{\delta}^{\frac{1}{2}\pi} \frac{\sin(2n+1)t}{t}\varphi(t)\,\mathrm{d}t. \tag{1.12}$$

We observe that the second term of (1.12) converges to 0 as $n \to \infty$ by the Riemann–Lebesgue theorem 1.3. In fact, $\varphi(t)/t$ is summable in $[\delta, \frac{1}{2}\pi]$.

Now, as $f(x)$ is of bounded variation near the point x which we are now considering, $f(x+0)$ and $f(x-0)$ exist. The value of $f(x)$ can therefore be changed to give

$$f(x) = \tfrac{1}{2}[f(x+0)+f(x-0)].$$

In this way, as the value of $f(x)$ is affected only at a point, there is no change in its Fourier series.

Our problem now is merely to prove that the first term of the right hand integrals of (1.12) can be arbitrarily small when we make δ sufficiently small irrespective of the value of n. By our hypothesis, we conclude:

(1) $\varphi(t)$ is of bounded variation in $[0, \delta]$;

(2) $\lim_{t \to +0} \varphi(t) = 0$.