I
A mathematical formulation of the Ricardian system*

Since his own time, David Ricardo has always occupied a privileged place among economists, even in periods when economic analysis has been developing along paths very different from the ones he pursued. It has never been easy, however, for Ricardo's many interpreters, to state his complete system in a rigorous and concise form, and the reason lies in the peculiarity of some of the concepts he used, which are not always defined in an unambiguous way. These concepts have encountered strong criticisms almost at any time, while – on the other hand – the bold analyses they made possible were exerting a sort of fascinating attraction.

In this essay, criticism is left aside and the more constructive approach is taken of stating explicitly the assumptions needed in order to eliminate the ambiguities. Then, the Ricardian system is shown to be very neat and even suitable for a mathematical formulation, with all the advantages of conciseness, rigour and clarity. The task is undertaken in sections 4 to 9, which form the main part of the paper (part II). To avoid digressions and lengthy references there, the difficulties Ricardo was faced with and the basic features of his theories are briefly reviewed in the first three introductory sections (part I).

I

1. *Theory of value*

The theory of value represents the most toilsome part of Ricardo's theoretical system and in our mathematical formulation it will entail the crudest assumptions. At the time it was put forward, the theory soon became the main target of the criticisms, which Ricardo tried to

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answer by re-writing twice (in the second and in the third edition) the chapter 'on value' of his *Principles*.[1] No fundamental change was introduced, however, and the three versions represent different ways of framing (in the light of the criticisms) a theory of value which remains essentially the same.[2]

The theory is fundamentally based on the cost of production measured in terms of quantity of labour. Utility[3] is considered to be absolutely essential to, but not a measure of, exchangeable value. To commodities which derive their value from 'scarcity alone'[4] (e.g., rare paintings) only a few words are devoted – they are not considered relevant for economic analysis; Ricardo is concerned only with commodities which are the outcome of a process of production. And of these commodities he is concerned with finding the 'primary and natural price', as against 'the accidental and temporary deviations of the actual or market price'.[5] He begins by restating Adam Smith's proposition that 'in the early stages of society, the exchangeable value of commodities . . . depends . . . on the comparative quantity of labour expended on each'.[6] Then, he takes a new and striking step by asserting that *the mentioned proposition is valid in general* and not only in the early stages of society, as Smith claimed. His argument may be roughly expressed in the following way. Suppose two commodities, *A* and *B*, the first of which requires the work of one worker for one year to be produced and the second the work of two workers for one year (the capital employed being just the amount of wages to be anticipated to the workers). Whatever the rate of profit may be, either 10 per cent or

[1] David Ricardo, *On the Principles of Political Economy and Taxation* (cit. as *Principles*). All references to Ricardo's works in this essay refer to the edition prepared by Piero Sraffa, *The Works and Correspondence of David Ricardo*, 11 vols, Cambridge, 1951–73, (cit. as *Works*).

[2] This is a view to which recently Mr Sraffa has given full support (*Works*, vol. I, Introduction, pp. xxxvii and ff.). Fragments of an early version of the Ricardian theory of value can be traced in Ricardo's early writings and in some letters (see the evidence given by Mr Sraffa, *Works*, vol. I, p. xxxi). It seems that Ricardo tried at the beginning to measure the relevant variables of his system in terms of a main agricultural commodity, namely corn, claiming that this commodity has the property of being both the capital and the product and, therefore, makes it possible to determine the ratio of profit to capital in physical terms, without any question of evaluation. This position was, however, very vulnerable and will not be considered in this essay, as Ricardo abandoned it before writing the *Principles*.

[3] Needless to say, the term 'utility' has for Ricardo, and in general for the Classics, a different meaning than for us to-day. It simply refers to the 'value in use' of a commodity as opposed to its 'value in exchange'. See *Principles*, p. 11.

[4] *Principles*, p. 12.

[5] *Principles*, p. 88.

[6] *Principles*, p. 12.

20 per cent or 30 per cent, the profit on the second commodity always is twice as much as on the first commodity; hence the relative price of the two goods always comes out as equal to the ratio of the quantities of labour required to obtain each of them.[7] If a 'commodity could be found, which now and at all times required precisely the same quantity of labour to produce it, that commodity would be of an unvarying value'[8]: it would be an *invariable standard* in terms of which the value of all commodities could be expressed.

This formulation of the theory, of course, did not remain unchallenged. Strong objections were immediately raised (by Malthus, McCulloch, Torrens and others) which may be summarized as follows. Let us suppose, returning to the mentioned example, that the production of commodity *B* requires the work of one worker for two years instead of the work of two workers for one year. In this case, Ricardo's principle no longer applies because, owing to the profits becoming themselves capital at the end of the first year, a change in the rate of profit *does* imply a change in the relative price of the two commodities, even though the relative quantities of labour required by them remain the same.[9] Ricardo could not ignore these objections and already in the first edition of the *Principles* he allowed for some exceptions to his general rule. All exceptions – as he later explained in a letter – 'come under one of time',[10] but he preferred discussing them, in the third edition of the *Principles*, under three groups (i. different proportions of fixed and circulating capital, ii. unequal durability of fixed capital, iii. unequal rapidity with which the circulating capital returns to its employer). However, while allowing for exceptions, Ricardo kept the fundamentals of his theory and tried to overcome the objections by appealing to the order of magnitude of the deviations caused by the exceptions, which he considered as responsible only for minor departures from his general rule. In the previous example, for instance, the modification introduced by the possibility that the same quantity of labour on *B* might be employed in one year or in two different years amounts simply to the effects caused by the capitalization of the profits calculated on the wages of the first year. Ricardo holds that this is a

[7] *Principles*, pp. 24 and ff.

[8] *Principles*, version of editions 1 and 2, see p. 17, footnote 3.

[9] An *invariable standard of value* presents therefore two distinct difficulties. First of all there is the difficulty of finding a commodity which 'now and at all times requires precisely the same quantity of labour to produce it'. Secondly, even if such a commodity were to be found, there is the further difficulty that its value would change with changes in the distribution of income.

[10] Letter to McCulloch, *Works*, vol. VIII, p. 193.

difference of minor importance.[11] Therefore, the conclusion is, the theory of value as stated in terms of quantities of labour, and independently of the distribution of income among the classes of the society, does hold, if not exactly, at least as a very good approximation ('the nearest approximation to truth'[12]). With this premise, Ricardo considers as 'the principal problem of Political Economy' that of determining 'the laws which regulate the distribution'.[13]

2. *Theory of distribution*

The participants in the process of production are grouped by Ricardo into three classes: landlords who provide land, capitalists[14] who provide capital and workers who provide labour. Total production is entirely determined by technical conditions but its division among the three classes – under the form of rent, profit and wages – is determined by the inter-action of many technical, economic and demographic factors. All Ricardo's analysis on this subject refers to what he calls the *natural* prices of rent, profits and wages. Divergencies of market prices from their natural level are considered only as temporary and unimportant deviations.

Rent, namely 'that portion of the produce of the earth which is paid to the landlords for the use of the original and indestructible power of the soil'[15] is determined by technical factors. The technical property that different pieces of land have different fertility and that successive applications of labour to the same quantity of land yield smaller and smaller amounts of product (law of diminishing returns) makes of rent

[11] *Principles*, pp. 36 and ff.

[12] Letter to Malthus, *Works*, vol. VIII, p. 279; see also Sraffa's Introduction, *Works*, vol. I, p. xl. With the acceptance of criticisms, between the first and the third edition of the *Principles*, also the choice of a 'standard of value' became more difficult. Ricardo reacted to the complication by changing his definitions. In the first edition of the *Principles* he regarded as 'standard' a commodity which would require at any time the same amount of unassisted labour (unassisted by capital); in the third edition he mentions a 'commodity produced with such proportions of the two kinds of capital [fixed and circulating] as approach nearest to the average quantity employed in the production of most commodities' (*Principles*, p. 63 and p. 45; see also *Works*, Introduction by Mr Sraffa, vol. I, p. xlii and ff.). Ricardo considered one year a good average and thought that perhaps gold could be the commodity that most closely approaches the requirement of an *invariable standard*. (*Principles*, p. 45.)

[13] *Principles*, p. 5.

[14] Ricardo calls them alternatively 'farmers' or 'manufacturers', according as he refers to agricultural or to industrial capitalists.

[15] *Principles*, p. 67.

a *net gain* for the landlords. Therefore, rent does not enter Ricardo's theory of value – it is a deduction from the total product. The value of commodities is determined by the quantity of labour employed on the marginal portion of land – that portion of land which yields no rent.

Wages are not related to the contribution of labour to the process of production, as in the modern theories they normally are. Like all economists of his time, Ricardo relates the level of wages to the physiological necessity of workers and their families to live and reproduce themselves. He is convinced that in any particular state of society there exists a *real* wage-rate (so to speak, a certain *basket of goods*) which can be considered as the 'natural price of labour'. It need not necessarily be at a strict *subsistence level*[16] (the minimum physiological necessities of life); but at that level which in a given country and in a given state of society, besides allowing workers to live, induces them to perpetuate themselves 'without either increase or diminution'.[17] When capitalists accumulate capital, demand for labour increases and the market wage-rate rises above its natural level. However, Ricardo believes that such a situation cannot be other than a temporary one because, as the conditions of workers become 'flourishing and happy', they 'rear a healthy and numerous family'[18] and the growth of population again brings back the real wage-rate to its *natural* level. It is very impressive to notice how strongly Ricardo is convinced of the operation of this mechanism. To be precise, he always speaks of a process which will operate 'ultimately' but the emphasis on it is so strong that his analysis is always carried on *as if* the response were almost immediate.

Profits, finally, represent a residual. Rent being determined by the produce of the marginal land put into cultivation, and the wage-rate by non-economic factors, what remains of the total production is retained, under the form of profit, by the capitalists, who are the organizers of the process of production. The capitalists are assumed to be always intent on moving their capital towards any sector of the economy that shows a tendency to yield a rate of profit above the average. This behaviour ensures the equalization of the rate of profit (after risk) all over the economy.

[16] 'The natural price of labour – Ricardo says – varies at different times, in the same country, and very materially differs in different countries. It essentially depends on the habits and customs of the people . . . Many of the conveniences – Ricardo adds – now enjoyed in an English cottage would have been thought luxuries in an earlier period of our history.' (*Principles*, pp. 96–7.)

[17] *Principles*, p. 93.

[18] *Principles*, p. 94.

3. *Theory of economic growth*

Economic growth is brought about essentially by the capitalists. The three classes in which Ricardo divides society have different peculiar characteristics. Landlords are considered as an 'unproductive class'[19] of wealthy people who become richer and richer, and consume almost all their incomes in *luxury-goods*. Workers also consume everything they get but in a different kind of goods – 'necessaries' – in order to live. Capitalists, on the other hand, are the *entrepreneurs* of the system. They represent the 'productive class'[20] of society. Very thrifty, they consume a small amount of what they obtain and devote their profits to capital accumulation.

The process of transforming profits into capital, however, cannot go on indefinitely. Owing to the diminishing returns of new capital (and labour) applied to the same quantity of land, or to less fertile lands, rent increases over time, in real and in money terms, the *money* wage-rate increases too[21], and consequently the profit-rate continuously falls.[22] When the rate of profit has fallen to zero, capitalists are prevented from accumulating any more; the growth process stops and the system reaches a *stationary state*. As a matter of fact – Ricardo adds – the stationary state will be reached *before* the extreme point where all profits have disappeared because, at a certain minimum rate of profit, the capitalists will lose any inducement to accumulate. The final outcome (the stationary state) is postponed in time by new inventions and discoveries, which increase the productivity of labour, but it is Ricardo's opinion that it will eventually be attained.

II

4. '*Natural*' *equilibrium in a two-commodity system*

It has been mentioned that Ricardo distinguishes two groups of commodities produced in the economy: 'necessaries' – or, we may call them wage-goods – and 'luxuries'. The most simple Ricardian system we can conceive of is, therefore, one where each of the two groups is reduced to one commodity. Let us begin with this case and make the following assumptions:

[19/20] *Principles*, p. 270.

[21] How this happens will appear very clearly in the mathematical treatment of the following sections.

[22] *Principles*, especially chapters VI and XXI.

(i) the system produces only one type of wage-good, let us call it corn;

(ii) to produce corn, it takes exactly one year;

(iii) capital consists entirely of the wage-bill; in other words, it is only circulating capital, which takes one year to be re-integrated;

(iv) there does exist an invariable standard of value, namely a commodity, let us call it gold – a luxury-good –, which at any time and place always requires the same quantity of labour to be produced. Its process of production also takes one year. Prices are expressed in terms of such a commodity and the monetary unit is that quantity of gold which is produced by the labour of one worker in one year.

The Ricardian system can now be stated in terms of equations. Taking the quantity of land in existence as given and supposing that its technical characteristics (fertility and possibilities of intensive exploitation) are known, the production of corn can be expressed by a technical production function, which we may assume to be continuously differentiable:

$$X_1 = f(N_1), \tag{1.1}$$

where: X_1 = physical quantity of corn produced in one year; N_1 = number of workers employed in the corn production;

with the following properties:

$$f(0) \geqslant 0, \tag{1.1a}$$

$$f'(0) > \bar{x}, \tag{1.1b}$$

where: $\bar{x}$ = natural wage-rate in terms of corn,

$$f''(N_1) < 0. \tag{1.1c}$$

The first inequality means that when no labour is employed, land is supposed to produce either something or nothing at all (negative production is excluded). The meaning of (1.1*b*) is that, at least when the economic system begins to operate and workers are employed on the most fertile piece of land, they must produce more than what is strictly necessary for their support, otherwise the whole economic system would never come into existence. Finally, (1.1*c*) expresses the law of diminishing returns.

The production function for gold is much simpler:

$$X_2 = \alpha N_2, \tag{1.2}$$

where: X_2 = physical quantity of gold produced in one year; N_2 = number of workers employed in the production of gold; α = physical quantity of gold produced by one worker in one year ($\alpha > 0$).

The following equations are self-explanatory:[23]

$$N = N_1 + N_2, \tag{1.3}$$

$$W = N x, \tag{1.4}$$

$$K = W, \tag{1.5}$$

$$R = f(N_1) - N_1 f'(N_1), \tag{1.6}$$

$$P_1 = X_1 - R - N_1 x, \tag{1.7}$$

where: N = total number of workers; N_1 = agricultural workers; N_2 = workers in the gold industry; W = total wage-bill, in terms of physical units of corn; x = real wage-rate (corn); K = physical stock of capital (corn); R = yearly rent, in real terms (corn); P_1 = yearly total profits, in real terms (corn), in the corn producing sector.

All variables introduced so far are in physical terms. Turning now to the determination of values, we have:

$$p_1 X_1 - p_1 R = N_1, \tag{1.8}$$

$$p_2 X_2 = N_2, \tag{1.9}$$

where: p_1 = price of corn; p_2 = price of gold.

Equations (1.8) and (1.9) are very important in the Ricardian system. They state that the value of the yearly product, *after deduction of rent*, is determined by the quantity of labour required to produce it. In our case, owing to the definition of the monetary unit, the value of the product, after paying rent, is exactly equal to the number of workers

[23] Equation (1.6) may not appear so evident as the other equations. Let me state, therefore, an alternative way of writing it. As explained in section 2, rent represents for Ricardo a net gain for the owners of the more fertile lands with respect to the owners of the marginal land (the land which yields no rent). Therefore, when N_1 workers are employed on land, the resulting total rent can be expressed as a sum of all the *net gains* for the non-marginal land-owners. In analytical terms:

$$R = f(0) + \int_0^{N_1} [f'(y) - f'(N_1)]\, dy, \tag{1.6a}$$

where $f(0)$, from (1.1*a*), is the produce that the land-owners can get from land without renting it, i.e. without any labour being employed. By solving the integral appearing in (1.6*a*), we obtain:

$$R = f(0) + f(N_1) - f(0) - N_1 f'(N_1),$$

which is exactly equation (1.6).

employed. From (I.1), (I.2) and (I.6), equations (I.8) and (I.9) may be also written:

$$p_1 = \frac{N_1}{X_1 - R} = \frac{1}{f'(N_1)}, \tag{I.8a}$$

$$p_2 = \frac{1}{\alpha}. \tag{I.9a}$$

Profits in the gold industry and total profits in the economy emerge as:

$$p_2 P_2 = p_2 X_2 - N_2 p_1 x, \tag{I.10}$$

$$\pi = p_1 X_1 + p_2 X_2 - p_1 R - p_1 W, \tag{I.11}$$

where: P_2 = profits, in terms of physical units of gold, in the gold industry; π = total profits, in terms of the standard of value.

After substituting from (I.1)–(I.10), equation (I.11) may be also written:

$$\pi = (N_1 + N_2)(1 - x p_1). \tag{I.11a}$$

At this point, the equations contain a theory of value and a theory of distribution but not yet a theory of expenditure. Since Ricardo assumes that all incomes are spent (Say's law), to determine the composition of total expenditure only one equation is necessary in the present model, specifying the production of one of the two commodities. Then the quantity produced of the other commodity turns out to be implicitly determined, as *total* production has already been functionally specified. The Ricardian theory is very primitive on this point. Workers are supposed to spend their income on necessities (corn, in our case) capitalists on capital accumulation (corn again, in our case) and land-owners on luxuries. Hence the determining equation is:[24]

$$p_2 X_2 = p_1 R. \tag{I.12}$$

Let us also write:

[24] To be precise, we should allow for a minimum of necessities to be bought by the land-owners. This *minimum*, however, introduces only a constant into the analysis without modifying its essential features. For simplicity, therefore, the procedure is followed of neglecting the constant, which amounts to considering the minimum as negligible and supposing that the whole rent is spent on luxuries. Similarly, a minimum of luxuries might be allowed to be bought by the capitalists. This *minimum* also will be considered as negligible.

$$w = p_1 x, \tag{1.13}$$

$$r = \frac{\pi}{p_1 K}, \tag{1.14}$$

where: w = monetary wage-rate; r = rate of profit.

So far 16 variables have appeared: X_1, X_2, N_1, N_2, N, W, x, K, R, P_1, P_2, π, p_1, p_2, w, r, but only 14 equations. Two more equations are needed in order to determine the system. In a situation which Ricardo considers as *natural*, the following two data have to be added:

$$x = \bar{x} > 0, \tag{1.15}$$

$$K = \bar{K} > 0, \tag{1.16}$$

where: $\bar{x}$ = *natural* real wage-rate, defined as that wage-rate which keeps population constant; $\bar{K}$ = given stock of capital at the beginning of the year.

The system is now complete and determinate.[25] It can be easily demonstrated (see the appendix) that properties (1.1*a*), (1.1*b*), (1.1*c*) and the inequalities put on (1.15)–(1.16) are sufficient conditions to ensure the existence and uniqueness of non-negative solutions. We may consider, therefore, the system of equations (1.1)–(1.16) as expressing the *natural* equilibrium of the Ricardian system.[26]

5. *Some characteristics of the Ricardian system*

Already at this stage, the system of the previous section clearly shows some of the most typical characteristics of the Ricardian model. First of all, it contains a theory of value which is completely and (owing to our explicit assumptions) rigorously independent of distribution. From equations (1.8*a*) and (1.9*a*), it appears that the value of commodities

[25] It may be interesting to notice that equations (1.1), (1.4), (1.5), (1.6), (1.7), (1.15) and (1.16), taken by themselves, form an extremely simplified but determined Ricardian system expressed in terms of corn, where any question of evaluation has not yet arisen, corn being the single commodity produced. This is the system which has been used by Mr Kaldor in his article 'Alternative Theories of Distribution', *The Review of Economic Studies*, 1955–6, pp. 83–100.

[26] To justify the terminology, let me mention that in his article 'On the Notion of Equilibrium and Disequilibrium', *The Review of Economic Studies*, 1935–6, pp. 100–5, Professor Ragnar Frisch distinguishes two types of equilibria: *stationary* and *moving*. The *natural* equilibrium of the Ricardian system is not a stationary one, as will be seen in a moment; it belongs to the *moving* type. Professor Frisch, in that article, describes a somewhat similar situation for the Wicksellian *natural* rate of interest.