PART I

MATHEMATICS

INTRODUCTION

There are many hundreds of sheets in the Portsmouth Collection relating to pure and applied mathematics, and much material on the controversy over the invention of fluxions and the differential calculus. In pure mathematics there are annotations made by Newton as a student from the authors he read, which led to his own earliest discoveries; fragmentary studies and propositions on various subjects; and drafts of more or less complete works, some of them later incorporated in Newton's published mathematical writings. These latter are more directly derived, however, from the lectures delivered between 1673 and 1683, copies of which are preserved in the Cambridge University Library. This enormous mass of historical material has been largely unexplored hitherto, and existing accounts of Newton's mathematical development have been based almost entirely on published sources.

Although our main interest in making selections for this volume has been in Newton as a theoretical physicist, it seemed appropriate to include one example of his early mathematical work. We have therefore chosen a reasonably complete and coherent tract, which is certainly early in date, and which has the further merit of being in English (No. 1, below). After this, we give an example of a draft in applied mathematics, which shows Newton using fluxions in solving the type of problem treated in the *Principia* (No. 2).

This is not the place to attempt a complete history of Newton's mathematical discoveries and methods based upon the manuscripts, an overdue work of scholarship which we must leave to others; we shall only sketch the development of some of his work. Newton began his more advanced mathematical study by reading Descartes' *Geometry* (1637) and William Oughtred's *Clavis Mathematicae* (1631), in the summer of 1664, and continued with the 'miscellanies' of Frans van Schooten[1] and the publications of John Wallis,[2] of which he made himself master during the winter of 1664–5. These are the authorities he names himself; he certainly read others and

[1] Presumably *Exercitationum Mathematicarum Libri quinque*, 1657.
[2] Chiefly *Arithmetica Infinitorum*, 1655.

profited as well from the geometrical teaching of Isaac Barrow, then Lucasian Professor. It is clear that Newton's reading as an undergraduate took him to the frontier of knowledge in mathematics, and also that his own original discoveries were a product of continuous development from what had been done before. The first of these discoveries seems to date from the early months of 1665 and resulted from his method of expressing quantities by means of infinite series: he discovered a method of drawing a tangent to a curve at any point, of determining the radius of curvature at any point, and of calculating an area bounded by a curve. By the summer of 1665 he had accomplished the quadrature of the hyperbola.

At first Newton used no algebraic concept or notation unfamiliar to contemporary mathematicians. He was relatively slow to formulate exactly the new concepts latent in his methods, and slower still to devise a new notation to express them, or a system involving them. His problem—one common to other contemporary mathematicians—was to handle continuously varying quantities; for example, in the parabola $y = ax^2$, the height of the ordinate y varies continuously with the value of x; moreover, the rate of change of y (for any given change in x) is dependent on the value of x, and the area enclosed under the curve is in turn dependent upon this rate of change. 'Rate of change' implies that the change takes place with respect to time, and indeed Newton thought of a curve as the line produced by a point moving with an arbitrary velocity, so that the co-ordinates x and y of the point also change with a corresponding and varying velocity, dependent upon the equation to the curve. Similarly, the algebraic expressions denoting the slope of the tangent to the curve at the point, the area enclosed under it and so forth have their appropriate velocities of change. Just as the distance moved in a straight line by a point is calculated as a product of velocities and times, so Newton calculated such algebraic expressions from their velocities of change, themselves derived from the rates of change of the co-ordinates.

His basic ideas about changing quantities may be formulated in his own first published description, from the *Principia*:

Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic sides and the like. These quantities

I here consider as variable and indetermined, and increasing or decreasing as it were by a continual motion or flow (*flux*): and I understand their momentary increments or decrements by the name of *moments*: so that the increments may be esteemed as added or positive *moments*, and the decrements as subtracted or negative ones. [Thus, if Y varies with X, and the *moment* of X be x and the corresponding *moment* of Y be y, the *moment* of the rectangle XY is $Xy+xY$, the small term xy being negligible.] But take care lest you regard *moments* as little finite quantities (*particulas finitas*). As soon as *moments* become finite quantities they cease to be *moments*. For to be finite is in some way contrary to their perpetual increase or decrease. *Moments* are to be understood as the just nascent beginnings of finite quantities.... It will be the same thing if, instead of *moments*, we use either the velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities), or any finite quantities proportional to these velocities.[1]

From the above it follows, for instance, that (if the *moment* of X be called x) the *moment* of aX^n is $anxX^{n-1}$: that is, x here corresponds to dX in Leibniz's notation for the calculus, and (since $[d(aX^n)]/dx = anX^{n-1}$), the *moment* of aX^n is $d(aX^n)\,dX$. The velocity of the increment or *moment* x therefore, which Newton later called its fluxion and denoted by $\dot{X}$, is dX/dt, but where the velocities of the *moments* are directly comparable, $\dot{v}$ can be taken as dv, and $\dot{y}$ as dy, etc.

Newton seems to have developed the method of fluxions during the summer of 1665, and to have begun illustrating it by application to particular problems in the autumn. The paper reproduced here (No. 1) was written in October 1666; and appears to be Newton's most complete exposition of his methods up to that time. It was briefly described by Brewster, but has never been printed.[2] It is written on twenty-four sides of eight sheets of paper, folded and stitched down the middle to make thirty-two pages. The handwriting and the peculiarities of spelling are typical of Newton's early manuscripts, but this one is less heavily corrected than many, no doubt because it was not his first attempt to draft these topics.

The tract opens with eight propositions introducing the study of equations: Newton considers the terms of an equation as geometrically formulated, being expressions of the relations

[1] Book II, Lemma ii, 250; Cajori, 249.
[2] Brewster, II, 12–14.

between lines described by points moving at different velocities for the same period of time. (The velocity is, in fact, the fluxion of the term.) The seventh proposition explains a method which is virtually differentiation; the eighth, the reverse process of integration.[1] Next (fol. 52v) Newton turns to the demonstration of these propositions, beginning with Proposition 7; Proposition 8 'is the converse of this seventh prop. and may be therefore analytically demonstrated by it'; next, Propositions 1 and 2 are proved. There are no demonstrations of Propositions 3–6. The rest of the paper consists of the application of the propositions to seventeen problems, each illustrated by numerous examples. The problems are:

1. To draw tangents to crooked lines.

2. To find the quantity of crookedness of lines.

3. To find the points distinguishing betwixt the concave and convex points of crooked lines.

4. To find the points at which lines are most or least crooked.

5. To find the nature of the crooked line whose area is expressed by any given equation.

6. The nature of any crooked line being given, to find other lines whose areas may be compared to the area of the given line.

7. The nature of any crooked line being given, to find its area, when it may be [done]. Or more generally, two crooked lines being given, to find the relation of their areas, when it may be [done].

[8. This number is omitted by mistake; Newton left a space, and deliberately altered the next number from 8 to 9.]

9. To find such crooked lines whose lengths may be found, and also to find their lengths.

10. Any curve line being given, to find other lines whose lengths may be compared to its length or its area, and to compare them.

11. To find curve lines whose areas shall be equal (or have any other given relation) to the length of any given curve line drawn into a given right line.[2]

12. To find the length of any given crooked line, when it may be done.

13. To find the nature of a crooked line whose length is expressed by any given equation (when it may be done).

After these purely mathematical problems there are five more concerned with centres of gravity, using the same methods

[1] The mathematical use of the words 'differentiate' and 'integrate' is of course post-Newtonian.

[2] Problem 11 is given after Problem 12.

as before.[1] They are preceded by two definitions and two lemmas.

[ ]. To find the centre of gravity in rectilinear plane figures.
[ ]. To find such plane figures which are equiponderate to a given plane figure in respect of an axis of gravity in any given position.
15. To find the gravity of any given plane in respect of any axis, given its position.
16. To find the areas of gravity of any planes.
17. To find the centre of gravity of any plane, when it may be [done].

For mathematical theory the seventh and eighth propositions are the most interesting and important. In the seventh proposition Newton shows how the 'relation of the velocities' (p, q, r) of varying quantities (x, y, z) may be obtained. p, q, and r are in fact the fluxions of x, y, and z, but the term fluxion is never used in this tract and moreover p, q, and r are usually stated as ratios. To derive the ratios containing p, q and r, Newton multiplies each term containing x^n by np/x, each term containing y^n by nq/y and so on, throughout the equation. Or as he otherwise expresses it, he multiplies the terms of the equation by the appropriate term in the series

$$\frac{3p}{x}, \frac{2p}{x}, \frac{p}{x}, 0, \quad -\frac{p}{x}, \quad -\frac{2p}{x}, \quad -\frac{3p}{x} \quad \text{or} \quad \frac{3q}{y}, \frac{2q}{y}, \frac{q}{y}, \ldots.$$

Then he adds the two or more products thus obtained, gaining a new equation which yields the relation of p, q, and r (or $\dot{x}$, $\dot{y}$ and $\dot{z}$).

For example, if the given equation were

$$x^4 + 2bx^3 + (b^2+2)\,x^2 + 2bx + 1 - y^2 = 0,$$

from the x-terms

$$4x^3p + 6bx^2p + (2b^2+4)\,xp + 2bp,$$

and the y-terms, $\quad -2yq,$

adding, $\quad p(4x^3 + 6bx^2 + (2b^2+4)\,x + 2b) = 2yq.$

Hence, $$\frac{q}{p} = \frac{2x^3 + 3bx^2 + (b^2+2)\,x + b}{y},$$

[1] These additional problems were not listed by Brewster.

from which it easily follows that $q/p = 2x+b$, which is the relation between $\dot{y}$ and $\dot{x}$ obtainable from the given equation.

More generally, however, Newton says that p, q, r, etc., are to be obtained by multiplying the terms by the series

$$\frac{ap+4bp}{x}, \quad \frac{ap+3bp}{x}, \quad \frac{ap+2bp}{x}, \quad \text{etc.,}$$

and proceeding as before, a and b signifying two numbers whether rational or irrational.

J. M. Child, who was apparently ignorant of Newton's manuscripts, considered that differentiation by this method (which follows the modern principles) was 'the firstfruits of independent work...the first great step that Newton took, working on his own original lines'.[1] He attributed this step to 1672; in fact, there is evidence throughout this tract that Newton used the method of differentiation by multiplication of the terms of an equation by the terms of a series in 1666. This tract thus shows that historians of mathematics who based their opinion entirely on printed sources have been unduly sceptical in their reaction to the accounts of Newton's mathematical discoveries written by those few nineteenth-century historians who consulted Newton's unpublished papers.

The reverse problem to differentiation, the 'inverse method of fluxions' or integration, is inevitably more complex. Could integration always be done, wrote Newton, all problems whatever might be resolved; but at least by the following rules it might be very often done. As he puts it, if an equation is given expressing the relation between x and the ratio p/q of the velocities of change of x and y (that is, of the fluxions $\dot{x}$ and $\dot{y}$), the problem is to find y. The first result is simple: if $q/p = ax^{m/n}$, then $y = [an/(m+n)]\,x^{(m+n)/n}$, which is the converse of the earlier rule for differentiation. Newton then discusses the integral of x^{-1}, which according to the rule should be $x^0/0$, a meaningless expression. He argues, rather obscurely, that in such a case y is a logarithm (or power) of x, and hence that if x is given, y may be found from a table of (natural) logarithms.

Newton then goes on to discuss the reduction of the fraction q/p, where the denominator of the expression has more than

[1] J. M. Child, 'Newton and the Art of Discovery', in W. J. Greenstreet, *Isaac Newton 1642–1727*, London, 1927, 125. (The equations quoted by Child are misprinted.)

one term, so that it can be integrated. He consolidates his methods in tables of integrals (fol. 50r and v). In explanation, apparently, of these latter arguments, he notes that the velocities

$$\begin{cases} a/(b+cz) \\ \surd(az+bzz) \\ \surd(a+bzz) \end{cases} \text{are to the integrals of} \begin{cases} a/(b+cz) \\ \surd(az+bzz) \\ \surd(a+bzz) \end{cases}$$

as the ordinate bc in a conic section is to the area enclosed by it under the curve, z representing the axis ab (Fig. 1). Such quadratures he has supposed to be known, since they are derivable from tables of logarithms and trigonometric functions. Then, finally, Newton says that all equations may be integrated mechanically, by forming them into a series (by division or extraction of roots or powers) and then integrating the successive terms. Thus,

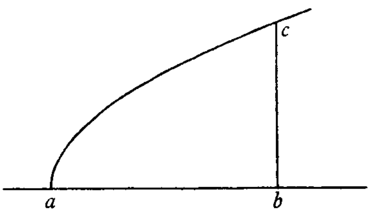

Fig. 1

$$\frac{a}{b+cx} = \frac{a}{b}\left(1-\frac{cx}{b}+\frac{c^2x^2}{b^2}-\frac{c^3x^3}{b^3}+\ldots\right),$$

of which the integral is

$$\frac{a}{b}\left(x-\frac{cx^2}{2b}+\frac{c^2x^3}{3b^2}-\frac{c^3x^4}{4b^3}+\ldots\right).$$

At this stage Newton introduces a special symbol for an integral (he has so far used none for the derivative). He gives the integral of $x^2/(ax+b)$ as

$$\frac{x^2}{2a}-\frac{bx}{a^2}+\Box\frac{b^2}{a^3x+a^2b}=y,$$

the last term signifying that part of the value of y which is correspondent to the same term in the expansion of $x^2/(ax+b)$ by division; that is, $\Box$ is an integral sign, equivalent to $\int$. This sign appears often in Newton's table of integrals: thus,

$$\Box\frac{c}{2b}\left(\frac{1}{a+bx^2}\right)=y$$

is the integral of $cx/(a+bx^2)$. The terms preceded by the symbol □ are those that have to be derived from tables, but Newton does not explain in detail how this was to be done, or how some of the integrals he lists are derived.

Newton's interpretation of a fluxion (fol. 52v)—though he does not give it that name—in the demonstration to Proposition 7 may be rendered as follows. Let x be the distance moved by a body in a finite time, and y that moved by a second body in the same time, y being a function of x. Let their velocities [that is fluxions] at any moment be p/q; then, though p and q are not constant, the ratio p/q is constant, because of the relation between x and y. Further, if the first body moves the infinitesimal distance po in the brief instant o,[1] and the second the distance qo, $(x+po)$ and $(y+qo)$ may be substituted for x and y in the former equation, $y = f(x)$, the ratio between these combined quantities being the same as the ratio between x and y. Hence, if the equation be the irrational one, $y = 2\surd(a+bx)/b$, we may write instead of

$$\frac{4a}{b^2}+\frac{4x}{b}-y^2 = 0,$$

$$\frac{4a}{b^2}+\frac{4x}{b}+\frac{4po}{b}-y^2-2yqo-q^2o^2 = 0.$$

Subtracting the known terms of the equation, dividing by o, and omitting any remaining terms in o, we have

$$\frac{4p}{b}-2yq = 0,$$

whence

$$\frac{q}{p} = \frac{2}{by}.$$[2]

This is the basic process for finding the velocity of a changing quantity, giving rise to the rules stated in Proposition 7; we should now write dx for p, and dt for o.

On fol. 55r, Newton introduces another new notation, referring to series of terms arranged in an equation. This

[1] o is, of course, not a cipher, but a letter.
[2] Putting dy for q, and dx for p, and substituting for y, gives the differential equation

$$\frac{dy}{dx} = \frac{1}{\surd(a+bx)}.$$

notation is not that of the later fluxional calculus. In it he writes

$\cdot\mathcal{X}$ for $f(x)$, where each term containing x has been multiplied by the power of x in it;

$\mathcal{X}\cdot$ for $f(x)$, where each term containing y has been multiplied by the power of y in it;

$:\mathcal{X}$ for $f(x)$, where each term containing x has been multiplied by $(2n+1)$, n being the power of x in it;

$\mathcal{X}:$ for $f(x)$, where each term containing y has been multiplied by $(2n+1)$, n being the power of y in it;

$\cdot\mathcal{X}\cdot$ for $f(x)$, where each term has been multiplied by $(n+1)$, n being the power of x or y in it.

Evidently he later decided that this was not a convenient notation, and abandoned it. In general, one may say that this tract shows that in 1666 Newton possessed methods of great power, but that these methods involved clumsy and laborious processes, especially in their application to well-known problems of the period, such as tangents and quadratures.

Later examples of Newton's use of fluxions in problems of pure mathematics are fairly common in the Portsmouth Collection; many occur in draft passages later reproduced in his mathematical works as printed during the eighteenth century. While drafts of *Principia* propositions in the synthetic geometrical form are frequently found, examples of the use of fluxions for solving such problems are rather rare. Yet it has long been understood that Newton tackled many problems by fluxional analysis before devising the synthetic proof of the truth of his solution. As early as 1715, in a review of the *Commercium Epistolicum* in the *Philosophical Transactions* inspired or perhaps actually written by Newton himself, it was alleged,

> By the help of the New Analysis Mr. Newton found out most of the Propositions in his *Principia Philosophiae* but because the ancients for making things certain admitted nothing into geometry before it was demonstrated synthetically, he demonstrated the propositions synthetically that the system of the heavens might be founded on good geometry. And this makes it now difficult for unskilful men to see the Analysis by which these propositions were found out.[1]

[1] *Phil. Trans.* 1715, 206. 'An Account of a Book called *Commercium Epistolicum*.'