TWO PAPERS ON REPRESENTATION THEORY

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These two papers are devoted to the representation theory of two infinite dimensional Lie groups, the group $SL_2(\mathbf{R})^X$ of continuous maps from a space X into $SL_2(\mathbf{R})$, and the group Diff(X) of diffeomorphisms (with compact support) of a smooth manifold X. Almost nothing of a systematic kind is known about the representations of infinite dimensional groups, and the mathematical interest of studying these very natural examples hardly needs pointing out.

Nevertheless the stimulus to the work came from physics, and I shall try to indicate briefly how the representations arise there. Physicists encountered not the groups but their Lie algebras, the algebra $\mathfrak{g}^X$ of maps from X to the Lie algebra $\mathfrak{g}$ of $SL_2(\mathbf{R})$, and the algebra Vect(X) of vector fields on X. The space X is physical space $\mathbf{R}^3$. Choosing a basis in $\mathfrak{g}$, to represent $\mathfrak{g}^X$ is to associate linearly to each real-valued function f on $\mathbf{R}^3$ three operators $J_i(f)$ $(i = 1, 2, 3)$, such that

$$[J_i(f), J_j(g)] = \sum_k c_{ijk} J_k(fg),$$

where c_{ijk} are the structural constants of $\mathfrak{g}$. In quantum field theory one writes $J_i(f)$ as $\int_{\mathbf{R}^3} f(x) j_i(x)\, dx$, where j_i is an operator-valued distribution.

Then the relations to be satisfied are

$$[j_i(x), j_j(y)] = \sum_k c_{ijk}\, \delta(x-y) j_k(y) \qquad (*)$$

where δ is the Dirac delta-function.

Similarly, to represent Vect($\mathbf{R}^3$) is to associate operators $P(f)$ to vector-valued functions f: $\mathbf{R}^3 \to \mathbf{R}^3$ so that $[P(f), P(g)] = P(h)$, where

$$h = \sum_i \left(f_i \frac{\partial g}{\partial x_i} - g_i \frac{\partial f}{\partial x_i} \right).$$

Writing $P(f) = \sum_i \int f_i(x) p_i(x) dx$ this becomes

$$[p_i(x), p_j(y)] = \delta_i(x-y)p_j(y) - \delta_j(x-y)p_i(x), \qquad (**)$$

where $\delta_k = \partial\delta/\partial x_k$.

Operators with the properties of $j_i(x)$ and $p_i(x)$ arise commonly in quantum field theory in the guise of "current algebras". For example, if one has a complex scalar field given by operators $\psi(x)$ (for $x \in \mathbf{R}^3$) which satisfy either commutation or anticommutation relations of the form $[\psi^*(x), \psi(y)]_\pm = \delta(x-y)$, then the "current-like" operators $p_i(x)$ defined by

$$p_i(x) = \frac{1}{2}\left\{ \psi^*(x) \frac{\partial\psi(x)}{\partial x_i} - \frac{\partial\psi^*(x)}{\partial x_i} \psi(x) \right\}$$

satisfy (**). Similarly if one has an N-component field ψ satisfying $[\psi^*_\alpha(x), \psi_\beta(y)]_\pm = \delta_{\alpha\beta}\delta(x-y)$, and $\sigma_1, \ldots, \sigma_n$ are $N \times N$ matrices representing the generators of a Lie algebra $\mathfrak{g}$ then the operators $j_i(x) = \psi^*(x)\sigma_i\psi(x)$ satisfy (*). (These examples are taken from [3].)

In connection with the quantization of gauge fields it is also worth mentioning that, as we shall see below, the most natural representation of the group of all smooth automorphisms of a fibre bundle is its action on $L^2(E)$, where E is the space of connections ("gauge fields") in the bundle, endowed with a Gaussian measure.

Representations of the group $SL(2, \mathbf{R})^X$.

This paper is concerned with the construction of a single irreducible unitary representation of the group G^X of continuous maps from a space X equipped with a measure into the group $G = SL_2(\mathbf{R})$. (In this introduction I shall always think of G as $SU_{1,1}$, i.e. as the complex matrices $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ such that $|a|^2 - |b|^2 = 1$.)

An obvious way of obtaining an irreducible representation of G^X is to choose some point x of X and some irreducible representation of G by operators $\{U_g\}_{g \in G}$ on a Hilbert space H, and to make G^X act on H through the evaluation-map at x, i.e. to make $f \in G^X$ act on H by $U_{f(x)}$. This representation can be regarded as analogous to a "delta-function" at x. More generally, for any finite set of points $x_1, \ldots, x_n$ in X and corresponding irreducible representations $g \mapsto U^{(i)}_g$ of G on Hilbert spaces $H_1, \ldots, H_n$ one can make G act irreducibly on the tensor product $H_1 \otimes \cdots \otimes H_n$ by assigning to $f \in G^X$ the operator $U^{(1)}_{f(x_1)} \otimes \cdots \otimes U^{(n)}_{f(x_n)}$. The object of the paper is to generalize this construction and produce a representation on a "continuous tensor product" of a family of Hilbert spaces $\{H_x\}$ indexed by the points of X (and weighted by the measure on X). There is a simple criterion for deciding

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whether a representation is an acceptable solution of the problem, in view of the following remark. For any representation U of G^X and any continuous map $\phi: X \to X$ there is a twisted representation ϕ^*U given by $(\phi^*U)_f = U_{f\phi}$. The representation to be constructed ought to have the property that ϕ^*U is equivalent to U whenever ϕ is a measure-preserving homeomorphism of X, i.e. for each such ϕ there should be a unitary operator T such that $U_{f\phi} = T_\phi U_f T_\phi^{-1}$.

The paper describes six different constructions of the representation, but only three are essentially different. Of these, one, described in §4 of the paper, is extremely simple, but not very illuminating because it is a construction a posteriori. I shall deal with it first. For any group Γ and any cyclic unitary representation of Γ on a Hilbert space H with cyclic vector $\xi \in H$ ("cyclic" means that the vectors $U_\gamma \xi$, for all $\gamma \in \Gamma$, span a dense subspace of H) one can reconstruct the Hilbert space and the representation from the complex-valued function $\gamma \mapsto \Psi(\gamma) = \langle \xi, U_\gamma \xi \rangle$ on Γ. To see this, consider the abstract vector space H_0 whose basis is a collection of formal symbols $U_\gamma \xi$ indexed by $\gamma \in \Gamma$. An inner product can be introduced in H_0 by prescribing it on the basis elements:

$$\langle U_{\gamma_1} \xi, U_{\gamma_2} \xi \rangle = \Psi(\gamma_1^{-1} \gamma_2).$$

The group Γ has an obvious natural action on H_0, preserving the inner product. Then H is simply the Hilbert space completion of H_0. The function Ψ is called the *spherical function* of the representation corresponding to $\xi \in H$.

In our case the group $\Gamma = G^X$ has an abelian subgroup K^X, where $K = SO_2$ is the maximal compact subgroup of G, and it turns out that the desired representation H contains (up to a scalar multiple) a unique unit vector ξ invariant under K^X. The corresponding spherical function is easy to describe. The orbit of ξ can be identified with G^X/K^X, i.e. with the maps of X into G/K, which is the Lobachevskii plane. (I shall always think of G/K as the open unit disk in $\mathbf{C}$ with the Poincaré metric.) Given two maps $f_1, f_2: X \to G/K$ the corresponding inner product is

$$\exp \int_X \log \operatorname{sech} \rho(f(x_1), f(x_2))\, dx,$$

where ρ is the G-invariant Lobachevskii or Poincaré metric on G/K. This means that the spherical function Ψ is given by

$$\Psi(f) = \exp \int_X \log \psi(f(x))\, dx,$$

where, if $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$, then $\psi(g) = |a|^{-1}$. To see that this construction does define a representation of G^X the only thing needing to be checked is that the inner product is positive. That is done in §4.2. But of course it is not clear from this point of view that the representation is irreducible.

A more illuminating construction of the representation is to realize the continuous tensor product as a limit of finite tensor products. To do this we actually represent the group of L^1 maps from X to G, i.e. the group obtained by completing the group of continuous maps in the L^1 metric (cf. §3.4). The L^1 maps contain as a dense subgroup the group of step-functions $X \to G$, and it is on the subgroup of step-functions that the representation is concretely defined.

If one is to form a limit from the tensor products of increasing numbers of vector spaces then the vector spaces must in some sense get "smaller". It happens that the group $SL_2(\mathbf{R})$ has the comparatively unusual property (cf. below) of possessing a family (called the "supplementary series") of irreducible representations H_λ (where $0 < \lambda < 1$) which do in a certain sense "tend to" the trivial one-dimensional representation as $\lambda \to 0$. Furthermore there is an isometric embedding $H_{\lambda+\mu} \to H_\lambda \otimes H_\mu$ whenever $\lambda + \mu < 1$. Now for any partition ν of X into parts $X_1, \ldots, X_n$ of measures $\lambda_1, \ldots, \lambda_n$ one can consider the group G_ν of those step-functions $X \to G$ which are constant on the steps X_i. The group G_ν acts on $\mathscr{H}_\nu = H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n}$; and when a partition ν' is a refinement of ν then $\mathscr{H}_\nu$ is naturally contained in $\mathscr{H}_{\nu'}$. Accordingly, the group $\bigcup_\nu G_\nu$ of all step-functions acts on $\bigcup_\nu \mathscr{H}_\nu$, and the desired representation is the completion of this.

The construction just outlined is carried out in §2 of the paper. A variant is described in §3, where the representations $\{H_\lambda\}$ of the supplementary series are replaced by another family $\{L_\lambda\}$ with analogous properties – the so-called "canonical" representations. These are cyclic but not irreducible, and L_λ contains H_λ as a summand. In terms of their spherical functions L_λ tends to H_λ as $\lambda \to 0$. The spherical function ψ_λ of L_λ is very simple, given by $\psi_\lambda(g) = |a|^{-\lambda}$ when $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$. In other words, L_λ is spanned by vectors ξ_u indexed by u in the unit disk G/K, and $\langle \xi_u, \xi_{u'} \rangle = \operatorname{sech}^\lambda \rho(u, u')$. Notice that the size of the generating G-orbit $\{\xi_u\}$ in L_λ tends to 0 as $\lambda \to 0$.

The remaining constructions exploit a quite different idea, which is useful in other situations too, as we shall see. I shall explain it in general terms.

Gaussian measures on affine spaces

Suppose that a group Γ has an affine action on a real vector space E with an inner product; i.e. to each $\gamma \in \Gamma$ there corresponds a transformation of E of the form $v \mapsto T(\gamma)v + \beta(\gamma)$, where $T(\gamma)\colon E \to E$ is linear and orthogonal, and $\beta(\gamma) \in E$. Then there is an induced unitary action of Γ on the space $L^2(E)$ of functions on E which are square-summable with respect to the standard Gaussian measure $e^{-\|v\|^2} dv$. Because this measure is not translation-invariant we have to define $U_\gamma : L^2(E) \to L^2(E)$ bv

$$(U_{\gamma^{-1}} f)(v) = \Phi_\gamma(v) f(T(\gamma)v + \beta(\gamma)),$$

where the factor

$$\Phi_\gamma(\nu) = e^{\frac{1}{2}\|v\|^2 - \frac{1}{2}\|T(\gamma)v + \beta(\gamma)\|^2} = e^{-\langle T(\gamma)v, \beta(\gamma)\rangle - \frac{1}{2}\|\beta(\gamma)\|^2}$$

is to achieve unitarity.

The importance of this construction is that the representation of Γ on $L^2(E)$ may be irreducible even when the underlying linear action on E by $\gamma \mapsto T(\gamma)$ is highly reducible. If the linear action T is given then the affine action is evidently described by the map $\beta: \Gamma \to E$. This is a "cocycle", i.e. $\beta(\gamma\gamma') = \beta(\gamma) + T(\gamma)\beta(\gamma')$, and it is easy to see that the affine space is precisely described up to isomorphism by the cohomology class of β in $H^1(\Gamma; E)$. One sometimes speaks of "twisting" the action of Γ on $L^2(E)$ by means of β.

Apart from the description just given there are two other useful ways of looking at $L^2(E)$. The first of these is as a "Fock space". For the Gaussian measure on E the polynomial functions are square-summable, and are dense in $L^2(E)$. So $L^2(E)$ can be identified with the Hilbert space completion of the symmetric algebra $S(E)$ of E. (A little care is necessary here: to make the natural inner product in $S(E)$ correspond to the Gaussian inner product in $L^2(E)$ one must identify $S^n(E)$ not with the homogeneous polynomials on E of degree n, but with the "generalized Hermite polynomials" of degree n.)

The other way of approaching $L^2(E)$ is to observe that it contains (and is spanned by) elements e^v for each $v \in E$, with the property that

$$\langle e^v, e^{v'} \rangle = e^{\langle v, v' \rangle}. \qquad \ldots (\dagger)$$

This means that $L^2(E)$ can be obtained from the abstract free vector space whose basis is a set of symbols $\{e^v\}_{v\in E}$ by completing it using the inner product defined by (†). Better still, one can start with symbols ϵ_v and define

$$\langle \epsilon_v, \epsilon_{v'} \rangle = e^{-\frac{1}{2}\|v - v'\|^2}:$$

this makes it plain that the construction uses only the *affine* structure of E. (Of course $\epsilon_v = e^{-\frac{1}{2}\|v\|^2} e^v$.)

The group $SU_{1,1}$ acts on the circle S^1, and has a very natural affine action on the space E_λ of smooth measures on S^1 with integral λ. E_λ is a coset of the vector space E_0 of smooth measures with integral 0. The invariant norm in E_0 is given by

$$\|\alpha\|^2 = \sum_{n>0} \frac{1}{n} |a_n|^2 \text{ when } \alpha = \sum_{n\neq 0} a_n e^{in\theta} d\theta.$$

Then $L^2(E_\lambda)$ is the "canonical representation" L_{λ^2} mentioned above. (This is stated, not quite precisely, as Theorem (7.1) of the paper.) For if α_u is the Dirichlet measure on S^1 corresponding to u in the unit disk G/K (i.e. α_u is the

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transform of $d\theta$ by any element of G which takes 0 to u) then $\lambda\alpha_u \in E_\lambda$ and

$$\frac{1}{2}\| \lambda\alpha_u - \lambda\alpha_{u'}\|^2 = \lambda^2 \log\cosh \rho(u, u').$$

Returning to the group we are studying, G^X has an affine action (pointwise) on E_1^X, the space of maps $X \to E_1$. (Notice that the linear action of G^X on E_0^X is highly reducible.) The space $L^2(E_1^X)$ has the appropriate multiplicative property with respect to X: for if X is a disjoint union $X = X_1 \cup X_2$ then $E_1^{X_1 \cup X_2} = E_1^{X_1} \times E_1^{X_2}$ and $L^2(E_1^{X_1 \cup X_2}) = L^2(E_1^{X_1}) \otimes L^2(E_1^{X_2})$. In the paper the equivalence of the representations on $L_2(E_1^X)$ and on the continuous tensor product of §§2 and 3 is proved by calculating the spherical functions, but it is quite easy to give an explicit embedding of the continuous tensor product in $L^2(E_1^X)$. For if Y is a part of X with measure λ then the map $E_1^Y \to E_{\sqrt{\lambda}}$ given by $f \mapsto \lambda^{-\frac{1}{2}} \int_Y f$ is compatible with the Gaussian measures, so that $L^2(E_{\sqrt{\lambda}})$ is a subspace of $L^2(E_1^Y)$, and for any partition $X = X_1 \cup \ldots \cup X_n$ with $m(X_i) = \lambda_i$ we have

$$\begin{aligned} L_{\lambda_1} \otimes \cdots \otimes L_{\lambda_n} &= L^2(E_{\sqrt{\lambda_1}}) \otimes \cdots \otimes L^2(E_{\sqrt{\lambda_n}}) \\ &\subset L^2(E_1^{X_1}) \otimes \cdots \otimes L^2(E_1^{X_n}) \\ &= L^2(E_1^X). \end{aligned}$$

Indeed it is pointed out in [10] that for any affine space E the space $L^2(E^X)$ can always be interpreted as a continuous tensor product of copies of $L^2(E)$ indexed by the points of X.

The irreducibility of the representation can be seen very easily in the Fock version. For the cocycle β vanishes on the abelian subgroup K^X, and so under K^X the representation breaks up into its components $S^n(E_0^X)$, on which K^X acts just by multiplication operators. The characters of K^X which arise are all distinct, so the irreducibility of the representation follows from the fact that the vacuum vector is cyclic, which is easily proved (cf. §5.2).

The three approaches to $L^2(E_1^X)$ are described in §§5, 6 and 7 of the paper. In connection with §6 notice that to give an affine action of Γ on a vector space E is the same thing as to give a linear action on a vector space H together with an invariant linear form $l: H \to \mathbf{R}$ such that $l^{-1}(0) = E$: the affine space is then $l^{-1}(1)$. (There is no point, in §6, in considering functions $f: X \to H$ other than those satisfying $l(f(x)) = 1$, and the formulae become less cumbersome under that assumption.)

§5 describes the Fock space version, but not quite in the standard form. The space E_1 of measures on the circle can be identified (by Fourier series) with a space of maps $\mathbf{Z} \to \mathbf{C}$. Accordingly E_1^X is a space of maps $X \times \mathbf{Z} \to \mathbf{C}$, and the symmetric power $S^k(E_1^X)$ is a space of maps

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$$\underbrace{X \times \ldots \times X}_{\leftarrow k \rightarrow} \times \underbrace{\mathbf{Z} \times \ldots \times \mathbf{Z}}_{\leftarrow k \rightarrow} \rightarrow \mathbf{C}$$

which are symmetric in the obvious sense. The effect of this point of view is to identify $L^2(E_1^X)$ with a space of functions on the free abelian group generated by the space X, i.e. on the space whose points are "virtual finite subsets" $\Sigma n_i x_i$ of X, with $n_i \in \mathbf{Z}$. This is intriguing, but whether it is more than a curiosity it is hard to say.

That concludes my account of the contents of the paper itself; but I shall mention some related matters. The most obvious question to ask is what class of groups G the method applies to. As it stands it evidently does not work for groups for which the trivial representation is isolated in the space of all irreducible representations. This excludes all compact groups, as for them the irreducible representations form a discrete set. The isolatedness of the trivial representation has been cleverly investigated by Kazhdan [5], who proved in particular that among semisimple groups the trivial representation is isolated if the group contains $SL_3(\mathbf{R})$ as a subgroup. The only simple groups not excluded by Kazhdan's criteria are $SO_{n,1}$ and $SU_{n,1}$ – notice that $PSL_2(\mathbf{R}) \cong SO_{2,1}$ and $PSL_2(\mathbf{C}) \cong SO_{3,1}$. For these the method works just as for $SL_2(\mathbf{R})$. (For example $SO_{n,1}$ is the group of all conformal transformations of S^{n-1}, and the affine space E_λ used above can be replaced by the space of measures on S^{n-1} with integral λ.)

A class of groups for which the trivial representation is not isolated consists of the semidirect products $G \tilde{\times} V$, where G is a compact group with an orthogonal action on a real vector space V. Indeed if Ω is a G-orbit in V an element $g \in G$ acts naturally on $L^2(\Omega)$, and $v \in V$ can be made to act by multiplication by the function $e^{i\langle v,\omega\rangle}$. When the compact orbit Ω is close to the origin in V the representation $L^2(\Omega)$ is close to the trivial representation (in the sense of its spherical function). Furthermore $G \tilde{\times} V$ has an obvious affine action on V: the induced action on $L^2(V)$ is the direct integral of the irreducible representations $L^2(\Omega)$ for all orbits $\Omega \subset V$.

Thus the methods of the paper apply to all groups of the form $G \tilde{\times} V$. The importance of this is that it provides a way of constructing a representation of the group $(G^X)_{sm}$ of *smooth* maps from a manifold X to a compact group G. For a smooth map $f: X \rightarrow G$ induces a map of tangent bundles $Tf: TX \rightarrow TG$, and this can be regarded as a map which to each point $x \in X$ assigns a "1-jet" $j(x) \in J_x\, G = G \tilde{\times} (T_x^*X \otimes \mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G. As the groups $J_x G$ is of the form $G \tilde{\times} V$ the method of the paper provides a representation of the group Γ of bundle maps $TX \rightarrow TG$. (The fact that $J_x G$ depends on x, giving rise to a bundle of groups on X, is not important.) Γ contains $(G^X)_{sm}$ as a subgroup, and it turns out that the representation constructed remains irreducible when restricted to $(G^X)_{sm}$, at least when $\dim(X) \geqslant 4$. That is proved in the papers [1] and [2].

It is interesting to notice that the group of bundle maps Γ is just the semidirect product $(G^X)_{sm} \tilde{\times} \Omega^1(X; \mathfrak{g})$, where $\Omega^1(X; \mathfrak{g})$ is the space of 1-forms on X with values in $\mathfrak{g}$; and the associated affine space E is the space of connections in the trivial G-bundle on X. The fact that the space of the representation of $(G^X)_{sm}$ is $L^2(E)$ is, of course, suggestive from the point of view of gauge theories in physics.

Representations of the group of diffeomorphisms

This paper is devoted to the representation theory of the group Diff(X) of diffeomorphisms with compact support of a smooth manifold X. (A diffeomorphism has compact support if it is the identity outside a compact region.)

The most obvious unitary representation of Diff(X) is its natural action on $H = L^2(X)$, the space of square-summable $\frac{1}{2}$-densities on X. (By choosing a smooth measure m on X one can identify $L^2(X)$ with the usual space of functions f on X which are square-summable with respect to m. Then the action of a diffeomorphism ψ on f will be $f \mapsto \tilde{f}$, where

$$\tilde{f}(x) = J_\psi(x)^{\frac{1}{2}} f(\psi^{-1} x)$$

and $J_\psi(x) = dm(\psi^{-1} x)/dm(x)$. But it is worth noticing that $L^2(X)$ is canonically associated to X, and does not involve m.)

From H a whole class of irreducible representations of Diff(X) can be obtained by the well-known method introduced by Weyl to construct the representations of the general linear groups. For any integer n the symmetric group S_n acts on the n-fold tensor product $H^{\otimes n} = H \otimes \cdots \otimes H$ by permuting the factors, and the action commutes with that of Diff(X). It turns out that under Diff(X) $\times$ S_n the tensor product decomposes

$$H^{\otimes n} = \bigotimes_\rho V^\rho \otimes W_\rho,$$

where $\{W_\rho\}$ is the family of all irreducible representations of S_n, and V^ρ is a certain irreducible representation of Diff(X). More explicitly, V^ρ is the space of L^2 functions $\underbrace{X \times \ldots \times X}_{n} \to W_\rho$ which are equivariant with respect to S_n: thus it makes sense even when ρ is not irreducible, and $V^{\rho \oplus \rho'} \cong V^\rho \oplus V^{\rho'}$. The class of representations $\{V^\rho\}$, which were first studied by Kirillov, is closed under the tensor product: if ρ and σ are representations of S_n and S_m then $V^\rho \otimes V^\sigma \cong V^{\rho \cdot \sigma}$, where $\rho \cdot \sigma$ is the representation of S_{n+m} induced from $\rho \otimes \sigma$. All of this is explained in §1 of the paper.

It is then natural to ask, especially when X is not compact, whether new representations of Diff(X) can be constructed by forming some kind of infinite tensor product $H^{\otimes \infty}$ and decomposing it under the infinite symmetric group S^∞ of all permutations of $\{1, 2, 3, \ldots\}$. This question is the main subject of the paper, and it is considered in the following way.

The L^2 functions $X^n \to W_\rho$ are the same as those $\tilde{X}^n \to W_\rho$ where

$\widetilde{X}^n \subset X^n$ is the space of n-triples of *distinct* points. The symmetric group S_n acts on $\widetilde{X}^n$, and the quotient space is $B_X^{(n)}$, the space of n-point subsets of X. Diff(X) acts transitively on $B_X^{(n)}$, and there is a unique class of quasi-invariant measures on it. The representation V^ρ can be regarded as the space of sections of a vector bundle on $B_X^{(n)}$ whose fibre is W_ρ. An appropriate infinite analogue of $B_X^{(n)}$ is the space Γ_X of infinite "configurations" in X, i.e. the space of countable subsets γ of X such that $\gamma \cap K$ is finite for every compact subset K of X. This space, and the probability measures on it, play an important role in both statistical mechanics and probability theory. One can imagine the points of a configuration as molecules of a gas filling X, or as faulty telephones.

Diff(X) does not act transitively on Γ_X: two configurations are in the same orbit only if they coincide outside a compact region. Nevertheless one can define (in many ways) measures on Γ_X which are quasi-invariant and ergodic under Diff(X). For each such measure μ there is an irreducible representation U_μ of Diff(X) on $L^2(\Gamma_X; \mu)$. More generally, for each representation ρ of a finite symmetric group S_n there is an irreducible representation U_μ^ρ: it is the space of sections of the infinite dimensional vector bundle on Γ_X whose fibre is the representation H^ρ of S^∞ induced from the representation $\rho \otimes 1$ of $S_n \times S_n^\infty$. (S_n^∞ denotes the subgroup of permutations in S^∞ which leave $1, 2, \ldots, n$ fixed.) More explicitly, one can consider a covering space $\Gamma_{X,n}$ of Γ_X defined by

$$\Gamma_{X,n} = \{(\gamma; x_1, \ldots, x_n) \in \Gamma_X \times X^n : x_i \in \gamma \text{ for } i = 1, \ldots, n\}.$$

$\Gamma_{X,n}$ is locally homeomorphic to Γ_X, and therefore a measure μ on Γ_X defines a measure $\widetilde{\mu}$, the "Campbell measure", on $\Gamma_{X,n}$. The space of the representation U_μ^ρ is the space of maps $\Gamma_{X,n} \to W_\rho$ which are S_n-equivariant and square-summable for $\widetilde{\mu}$.

The simplest and most important measures on Γ_X are the Poisson measures μ_λ (parametrized by $\lambda > 0$), for which the measure of the set $\{\gamma \in \Gamma_X : \mathrm{card}(\gamma \cap K) = n\}$ is $\left(\frac{\lambda m}{n!}\right)^n e^{-\lambda m}$, where m is the measure of K. More can be said about the representations $U_\lambda^\rho = U_{\mu_\lambda}^\rho$ in the Poisson case:

(i) They form a closed family under the tensor product, and have the following simple behaviour

(a) $U_\lambda^\rho \cong U_\lambda \otimes V^\rho$, and

(b) $U_\lambda \otimes U_{\lambda'} = U_{\lambda+\lambda'}$.

(ii) U_λ is what is called in statistical mechanics an "N/V limit". In other words, if X is the union of an expanding sequence $X_1 \subset X_2 \subset X_3 \subset \ldots$ of open relatively compact submanifolds such that X_N has volume $\lambda^{-1}N$ then $L^2(\Gamma_X; \mu_\lambda)$ is the limit as $N \to \infty$ of the spaces $L^2_{\mathrm{sym}}((X_N)^N)$ of symmetric L^2 functions of N points in X_N. (This is explained in [4], [7], [8].)

(iii) U_λ has a more concrete realization as $L^2(E_\lambda)$, where E_λ is an affine space with a Gaussian measure (and an affine action of Diff(X)). E_λ is the space of $\frac{1}{2}$-densities f on X which are close to the standard Lebesgue $\frac{1}{2}$-density $f_\lambda = (\lambda\, dx)^{\frac{1}{2}}$ as $x \to \infty$, in the sense that $f - f_\lambda$ belongs to $H = L^2(X)$. This is an affine space associated to the vector space H and the cocycle β: Diff$(X) \to H$ given by

$$\beta(\psi) = \lambda^{\frac{1}{2}}(J_\psi^{\frac{1}{2}} - 1),$$

where $J_\psi(x) = dm(\psi^{-1}x)/dm(x)$ as before. As we have seen when discussing the representations of G^X, $L^2(E_\lambda)$ can also be regarded as a Fock space $S(H) = \bigotimes_{n \geqslant 0} L^2_{\text{sym}}(X^n)$, but with the natural action of Diff(X) twisted by the cocycle β. Because β vanishes on the subgroup Diff(X, m) of measure-preserving diffeomorphisms we see that in the Poisson case the representations associated to infinite configurations break up and give us nothing new when restricted to Diff(X, m).

In the paper the equivalence of $L^2(E_\lambda)$ and $L^2(\Gamma_X)$ is proved by considering the spherical functions, but it can also be described explicitly as a sequence of maps $L^2_{\text{sym}}(X^n) \to L(\Gamma_X)$. In fact $L^2(X) \to L(\Gamma_X)$ takes $\lambda^{\frac{1}{2}} f$ to the function

$$\gamma \mapsto \sum_{X \in \gamma} f(x) - \lambda \int_X f(x)dx,$$

while $L^2_{\text{sym}}(X \times X) \to L^2(\Gamma_X)$ takes λf to

$$\gamma \mapsto \sum_{x, y \in \gamma} f(x, y) - 2\lambda \sum_{x \in \gamma} \int_X f(x, y)dy + \lambda^2 \int\int_{X \times X} f(x, y)dxdy,$$

and so on.

The fact that there is a Gaussian realization of the representation is closely connected with the property of the Poisson measure μ_λ called "infinite divisibility". The latter means that if X is the disjoint union of two pieces X_1 and X_2, so that $\Gamma_X = \Gamma_{X_1} \times \Gamma_{X_2}$ up to sets of measure zero, then $\mu_\lambda = \mu_\lambda^{(1)} \times \mu_\lambda^{(2)}$, where $\mu_\lambda^{(i)}$ is the projection of μ_λ on Γ_{X_i}. This implies that when the representation U_λ of Diff(X) is restricted to the subgroup Diff$(X_1) \times$ Diff(X_2) it becomes $U_\lambda^{(X_1)} \otimes U_\lambda^{(X_2)}$, a property which must certainly be possessed by a construction of the type of $L^2(E_\lambda)$.

The reader may at first be confused by the fact that the affine action on $L^2(E_\lambda)$ used in this paper is the Fourier transform of the natural one used in the paper on G^X. Perhaps it is worth pointing out explicitly that if a group G acts orthogonally on a real vector space H with an inner product, and β: $G \to H$ is a cocycle, and $L^2(H)$ is formed using the standard Gaussian measure, then the following two unitary actions of G on $L^2(H)$ are unitarily equivalent

(a) $g \mapsto A_g$, where $(A_g\phi)(h) = e^{\frac{1}{2}\|h\|^2 - \frac{1}{2}\|h - \beta(g)\|^2}\, \phi(g^{-1}(h - \beta(g)))$,