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Introduction to the Representation Theory of Compact and Locally Compact Groups

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PART II : REPRESENTATIONS OF LOCALLY COMPACT GROUPS

FOREWORD

These are notes from a graduate course given in Lausanne (Switzerland) during the winter term 1978-79 (Convention romande des enseignements de 3^e cycle en mathématiques).

This term was devoted to a self-contained approach to representation theory for locally compact groups, using only *integral methods*. The sole prerequisite was a basic familiarity with the theory for finite groups (e.g. as contained in the first chapter of Serre 1967). For didactic reasons, I spent the first half of the term discussing compact groups, trying to be more elementary and more complete in this part. In particular, I have given several proofs of the main results. For example, the Peter-Weyl theorem is proved first with the use of the Stone-Weierstrass approximation theorem (p.33) and then without it (p.36). The "finiteness theorem" (irreducible representations of compact groups are finite dimensional) is proved first for Banach (or barrelled) spaces ((5.8)p.46), then in the general case (quasi-complete spaces) ((7.9)p.69) and finally in a more elementary fashion for Hilbert spaces ((8.5)p.81) . Thus I hope that readers with various backgrounds will be able to benefit from these notes.

My way of introducing the subject has forced me to repeat some definitions in the second part where I gradually assume more from my reader (this is particularly so as far as measure theory is concerned). This part in no way claims to be complete and only has an introductory purpose. I consider that the existing books on the subject more than prove that complete treatises on the subject are heavy going...

Finally, I should mention that the representation theory for Lie groups - starting with the rotation group $SO_3(\mathbb{R})$ and compact Lie groups - was considered in subsequent graduate courses, but has not been included in these notes since *differential methods* in the representation theory of Lie groups are probably more readily available in recent texts.

My presentation of the subject has certainly been influenced by R. Godement whose courses introduced me to this field. I would like to thank him here.

I would also like to thank Sylvie Griener who read the manuscript and helped me to detect various inaccuracies, and my wife Ann for hints on language.

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CONVENTIONAL NOTATIONS AND TERMINOLOGY

$\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ fundamental numerical sets
$\mathbb{H}$ real quaternions (skew-field, $\dim_{\mathbb{R}}(\mathbb{H}) = 4$, $\mathbb{R}$-basis $1,i,j,k$)
$\mathbb{F}_q$ finite field with q elements ($q = p^f$ for some prime p)
$\mathbb{Q}_p$ field of p-adic numbers, $\mathbb{Q}_p \supset \mathbb{Z}_p$ ring of p-adic integers
$A^\times$ group of units in a ring A ($k^\times = k - \{0\}$ if k is a field)
$\mathbb{R}_+^\times$ multiplicative group of positive real numbers $x > 0$
(= neutral connected component of $\mathbb{R}^\times$)
ϕ empty set, Card(X) = number of elements of X
countable: finite or denumerable (equipotent to some part of $\mathbb{N}$)
$E \hookrightarrow F$ injective (one-one into) map
χ_A characteristic function of a subset $A \subset X$: = 1 on A, = 0 outside A
$f|_A$ restriction of a mapping to a subset $A \subset X$
C(X,E) space of continuous maps $f : X \to E$
$C(X) = C(X,\mathbb{C})$, $C_{\mathbb{R}}(X) = C(X,\mathbb{R})$ continuous numerical functions on X
$C_c(X)$ subspace of C(X) consisting of functions with compact support
Sup ess $|f|$: smallest M with $|f| \leqslant M$ nearly everywhere (on a measured space)
$\check{f}(x) = f(x^{-1})$ symmetric function on a group G
$f^*(s) = \overline{f(s^{-1})}$ on a unimodular group (cf. (14.2) p.133 in general)

scalar products are always linear in the *second* factor
$a(x \mid y) = (x \mid ay) = (\bar{a}x \mid y)$
normal subgroup = invariant subgroup (= distinguished subgroup)
commutative = abelian (= Abelian!)
in sec. 20-21, separable group means locally compact group admitting
a countable basis for the open sets (hence has a countable
dense subset)
$1_n = id_n =$ id. unit matrix (in dim n)
${}^tg$, ${}^tA$ transpose of a matrix ($A^* = {}^t\bar{A}$, $\check{g} = {}^tg^{-1}$ contragredient)
Π_n vector space of (complex) polynomials in z and degree $\leqslant n$ (p.102)
X^G set of fixed points in a group action of G on X
Hausdorff space = T_2-space (= separated space)