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PART I :

REPRESENTATIONS OF COMPACT GROUPS

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1 COMPACT GROUPS AND HAAR MEASURES

Before starting representation theory, it is certainly appropriate to start with a review of examples of compact groups.

EXAMPLES

Some matrix groups

Let $O_n(\mathbb{R})$ denote the group of real $n\times n$ matrices which preserve the standard quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2$. Elements of this group are real $n\times n$ matrices g satisfying the relation ${}^t g\, g = 1_n$ (the columns of the matrix g must constitute an orthonormal basis of $\mathbb{R}^n$). This group is the *orthogonal group* : it is a compact subgroup of the general linear group $Gl_n(\mathbb{R})$ in n real variables. Since the relation ${}^t g\, g = 1_n$ implies $(\det g)^2 = 1$ hence $\det g = \pm 1$ and both cases occur in $O_n(\mathbb{R})$, we see that this group is not connected. Its index two subgroup

$$SO_n(\mathbb{R}) = O_n(\mathbb{R}) \cap Sl_n(\mathbb{R}) \quad \textit{special orthogonal group}$$

is known to be connected (Chevalley 1946, Dieudonné 1970 (16.11.7) p.68). The first non-trivial group in this series is the *circle group*

$$\mathbb{R}/\mathbb{Z} \longrightarrow SO_2(\mathbb{R}) \quad \text{(isomorphism)}$$

$$t \bmod \mathbb{Z} \longmapsto e^{2\pi i t} = a + ib \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ (a^2 + b^2 = 1) .$$

The next one is the *rotation group* $SO_3(\mathbb{R})$ (we shall study it in detail). For $n \geqslant 3$, the (special) orthogonal groups $SO_n(\mathbb{R})$ are not commutative.

We can use the complex field $\mathbb{C}$ (instead of $\mathbb{R}$) and consider the hermitian form $\bar{z}_1 z_1 + \bar{z}_2 z_2 + \ldots + \bar{z}_n z_n$ on $\mathbb{C}^n$, thus defining *unitary transformations* g as complex $n\times n$ matrices satisfying $g^* g = 1_n$ (recall that $g^* = {}^t\bar{g}$). The *unitary group* $U_n(\mathbb{C})$ is a compact connected (loc. cit.) subgroup of $Gl_n(\mathbb{C})$. The first such group is the circle group $U_1(\mathbb{C})$ (identified with the multiplicative group of complex numbers of modulus 1). We shall see later that the quotient of $U_2(\mathbb{C})$ by its center is (isomorphic to) $SO_3(\mathbb{R})$. Quite generally, the circle group $U_1(\mathbb{C})$ can be embedded *diagonally* into $U_n(\mathbb{C})$, the image of this embedding $U_1(\mathbb{C}) \hookrightarrow U_n(\mathbb{C})$

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being the center of the unitary group $U_n(\mathbb{C})$. Thus the center of $U_n(\mathbb{C})$ is connected (but the center of $O_n(\mathbb{R})$ is finite). Imposing furthermore the determinant 1 condition, we define the *special unitary groups*

$$SU_n(\mathbb{C}) = U_n(\mathbb{C}) \cap Sl_n(\mathbb{C}) .$$

(The condition $g^* g = 1_n$ in $U_n(\mathbb{C})$ implies $|\det g|^2 = 1$ and all complex numbers having modulus 1 are determinants of elements of $U_n(\mathbb{C})$.)
The center of $SU_n(\mathbb{C})$ consists of the scalar matrices of determinant 1 : it is a cyclic group of order n isomorphic to the subgroup of n^{th} roots of 1 (in $\mathbb{C}^\times$ or $U_1(\mathbb{C})$).

Similar considerations hold over the field $\mathbb{H}$ of real quaternions (the involution $q \mapsto \bar{q}$ being the quaternionic conjugation) with respect to the real bilinear form $\bar{q}_1 q_1 + \bar{q}_2 q_2 + \ldots + \bar{q}_n q_n$. But since this field $\mathbb{H}$ is not commutative, some care has to be taken with respect to the representation of $\mathbb{H}$-linear mappings from $\mathbb{H}^n$ into itself by quaternionic $n \times n$ matrices (one should say *left* $\mathbb{H}$-linear mappings to be quite precise). Thus one can construct compact connected groups $U_n(\mathbb{H}) \supset SU_n(\mathbb{H})$, which are also called *symplectic* groups.

The three *series* $SO_n(\mathbb{R})$ $(n \geqslant 3)$, $SU_n(\mathbb{C})$ $(n \geqslant 2)$ and $SU_n(\mathbb{H})$ $(n \geqslant 1)$ are the *classical groups*. Together with five *exceptional* groups, they exhaust the list of compact connected "simple" groups (more precisely, their center is finite and the quotient by their center is simple).

Some connected groups (not Lie groups)

The simplest example of a compact connected group which is not a Lie group is certainly the group $G = (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ (infinite product of circle groups). This group is commutative and each neighbourhood of its neutral element 0 contains a subgroup

$$G_n = \{0\} \times (\mathbb{R}/\mathbb{Z})^{[n,\infty[}$$

(this follows immediately from the definition of the product topology). Thus G contains *arbitrarily small subgroups*. More generally, let $(G_i)_I$ be a family of (non-trivial) compact connected groups. The product $G = \prod G_i$ is also a compact and connected group. Since a topological group is *metrizable* exactly when there is a countable fundamental system of neighbourhoods of its neutral element, we see that such products are

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metrizable precisely when the family I is countable. For compact groups, the following properties are equivalent (cf. also (5.11))

i) G is metrizable,

ii) G has a countable basis for open sets.

They imply

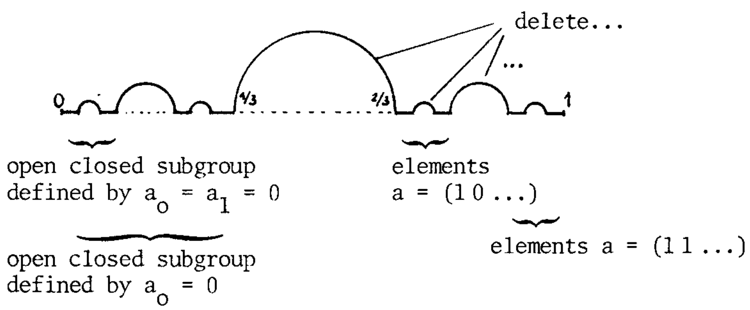
iii) G is separable (there is a countable family which is everywhere dense in G).

Totally discontinuous groups

In any topological group G, the connected component of the neutral element G^o is a closed and normal subgroup. When $G^o = \{e\}$, the only connected subsets of G are the points (and the empty set!) and G is *totally discontinuous*. In general, G/G^o will be totally discontinuous. A locally compact space which is totally discontinuous has the following property : each point has a fundamental system of open and closed neighbourhoods (Bourbaki 1971, TG II cor. of prop.6 p.32). The simplest example of totally discontinuous compact group is the *Cantor group*

$$G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} .$$

Its elements are the infinite sequences $a = (a_n)$ with $a_n = 0$ or 1 $(n \in \mathbb{N})$. The topological space underlying this group is usually obtained by removing successively from the unit interval $I = [0,1]$, its middle third $]1/3,2/3[$, then removing from each remaining interval its (open) third, etc... (cf. picture below). The intersection of this decreasing family of compact sets is the Cantor set. The topology induced by the real line coïncides with the product topology when elements of this set are represented in the dyadic system (as usual).



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The Cantor group is commutative and is not the most interesting totally discontinuous group. More sophisticated examples of totally discontinuous groups occur naturally in two contexts.

p-adic groups. The topological ring of p-adic integers $\mathbb{Z}_p$ can be defined either as (here, p is any prime number)

completion $\widehat{\mathbb{Z}_{(p)}}$ of the local ring $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ (consisting of fractions a/b with b not divisible by p),

or as

inverse limit $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ (with respect to the canonical homomorphisms $\mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ of finite rings).

The second definition makes obvious the fact that $\mathbb{Z}_p$ is compact. The additive group $\mathbb{Z}_p$ and the multiplicative group $\mathbb{Z}_p^\times$ are compact commutative groups. The groups

$$Gl_n(\mathbb{Z}_p) : \text{p-adic } n \times n \text{ matrices } g \text{ with } \det g \in \mathbb{Z}_p^\times$$

are good examples of totally discontinuous compact groups (for $n \geqslant 2$, these groups are not commutative).

Galois groups. Let k be any field, k_s denoting a *separable closure* of k (in some algebraic closure $\bar{k}$ of k). The group

$$G = Gal(k_s/k) = Aut_k(k_s)$$

is a compact topological group with respect to the *Krull topology*. Recall that this topology on G is defined by taking for fundamental system of neighbourhoods of the neutral element of G (the identity of k_s) the subgroups $G' = Gal(k_s/k')$ for all *finite* (separable) subextensions k' of k. These subgroups G' are both open and closed so that G is totally discontinuous. Moreover, with this topology, there is a Galois correspondence (inclusion reversing)

$$\left\{\begin{matrix}\text{closed subgroups}\\ \text{of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{intermediate extensions}\\ \text{between } k \text{ and } k'\end{matrix}\right\}$$

$$H = Gal(k_s/L) \longleftrightarrow L = \text{Fixed field under } H \quad .$$

Compact totally discontinuous groups are always pro-finite groups.

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HAAR MEASURE

The main analytical tool in the study of compact groups is the Haar integral. We shall construct this integral (or Radon measure) for continuous functions only, assuming that the reader is familiar with the extension procedure to Borel functions and sets (and eventually to measurable sets). However, very little of the abstract integration theory is needed : L^p spaces can be defined abstractly as suitable completions of the space of continuous fonctions ($1 \leq p < \infty$, p = 1 and 2 being the most important cases). Negligible sets can be defined as those sets N which have the following property : for each $\varepsilon > 0$, there is a continuous positive function $f = f_\varepsilon$ which has a restriction to N, $f_N \geq 1$ and integral smaller than ε. Here is the main statement.

Theorem. Let G be a compact group. Then, there is a unique linear form

$m : C(G) \longrightarrow \mathbb{C}$ (C(G) : space of continuous maps $G \longrightarrow \mathbb{C}$)

having the properties

1. $m(f) \geq 0$ for $f \geq 0$ (m is *positive*) ,
2. $m(1) = 1$ (m is *normalized*) ,
3. $m({}_sf) = m(f)$ for ${}_sf(g) = f(s^{-1}g)$ ($s, g \in G$)
   (m is *left invariant*) .

Moreover, this linear form m is also *right invariant*

4. $m(f_s) = m(f)$ for $f_s(g) = f(gs)$ ($s, g \in G$) .

The proof of this theorem can be based on the following two classical results.

a) *(Ascoli's theorem) Let X be a compact topological space, E a Banach space and Φ a subset of $C_E(X) = C(X;E)$ (Banach space of continuous mappings $X \longrightarrow E$ with the uniform norm). Then Φ is relatively compact (i.e. has a compact closure in $C_E(X)$) if and only if Φ is equicontinuous and all sets $\Phi(x) = \{f(x) : f \in \Phi\}$ ($x \in X$) are relatively compact in E.*

(cf. Dieudonné 1960, p.137 for the metric case, or Rudin 1973, p.369 .)

b) *(Kakutani's fixed point theorem) Let E be a Banach space (or any locally convex topological vector space), K a convex $\neq \emptyset$ compact subset of E, and G a compact group acting linearly on E. If the action $\lambda : G \longrightarrow Gl(E)$ leaves K invariant*

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$\lambda(g)K \subset K$ ($g \in G$) and $\Phi = \lambda(G)$ is equicontinuous, then there is a fixed point of G in K.

(cf. Rudin 1973, p.120 .)

Here is a construction of the measure m. For $f \in C(G)$ (i.e. f is a continuous complex valued function on G) we denote by C_f the convex hull of all left translates ${}_sf$ ($s \in G$) of f. Thus, the elements of C_f are the functions which are finite sums of the form

$$\sum_{\text{finite}} a_i\, f(s_i x) \quad \text{with} \quad a_i > 0 \text{ and } \Sigma a_i = 1 \quad .$$

Clearly if $g \in C_f$

$$\|g\| = \operatorname{Max}_G |g(x)| \leqslant \operatorname{Max}_G |f(x)| = \|f\| \quad .$$

In particular, all sets $C_f(x) = \{g(x) : g \in C_f\}$ are bounded and relatively compact in $\mathbb{C}$. On the other hand, since G is compact, f is *uniformly* continuous on G : for each $\varepsilon > 0$, there exists a neighbourhood $V = V_\varepsilon$ of the neutral element $e \in G$ with the property

$$y^{-1}x \in V \implies |f(y) - f(x)| < \varepsilon \, .$$

Since $(s^{-1}y)^{-1}s^{-1}x = y^{-1}ss^{-1}x = y^{-1}x$, we shall also have

$$|{}_sf(y) - {}_sf(x)| < \varepsilon \quad \text{as soon as} \quad y^{-1}x \in V \quad .$$

Making convex combinations of these, we also infer

$$|g(y) - g(x)| < \varepsilon \quad \text{as soon as} \quad y^{-1}x \in V \quad (\text{all } g \in C_f).$$

This proves that the set C_f is uniformly equicontinuous. By Ascoli's theorem, we conclude that C_f is relatively compact in C(G). Let K_f denote the closure of C_f in C(G): this is a compact convex set. The compact group G acts by left translations (isometrically) on C(G) and leaves C_f hence also $K_f = \bar{C}_f$ invariant. Kakutani's theorem asserts then that there is a fixed point of this action of G in K_f. Such a fixed point is a constant function

$${}_sg = g \quad (s \in G) \implies g(s^{-1}) = {}_sg(e) = g(e) = c \quad (s \in G).$$

Such a constant has the property of being approximated by elements of C_f i.e. convex combinations of left translates of f :

for each $\varepsilon > 0$, there are finitely many $s_i \in G$ and $a_i > 0$ with

$$\Sigma a_i = 1 \quad \text{and} \quad |c - \Sigma a_i f(s_i x)| < \varepsilon \quad (x \in G) \qquad (1).$$

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Let us show that there is only one constant function in K_f. For this, we start the same construction with *right* translations of f (we can apply the preceding construction to the opposite group G' of G, or the function $f'(x) = f(x^{-1})$ ...), obtaining a relatively compact convex set C'_f with compact convex closure K'_f containing a constant function c'. It will be enough to show c = c' (all constants c in K_f must be equal to *one chosen constant* c' of K'_f, and conversely!). There is certainly a finite combination of right translates which is close to c'

$$\left| c' - \sum b_j f(xt_j) \right| < \varepsilon \quad (\text{for some } t_j \in G,\ b_j > 0 \text{ with } \sum b_j = 1).$$

Let us multiply this equality by a_i and put $x = s_i$

$$\left| c' a_i - \sum a_i b_j f(s_i t_j) \right| < \varepsilon a_i \tag{2}$$

Summing over i, we obtain

$$\left| c' \sum a_i - \sum_{i,j} a_i b_j f(s_i t_j) \right| < \varepsilon \sum a_i = \varepsilon \tag{3}$$

Operating symmetrically on (1) (multiplying by b_j, putting $x = t_j$ and summing over j), we find

$$\left| c' - \sum_{i,j} a_i b_j f(s_i t_j) \right| < \varepsilon \tag{4}$$

Adding (or subtracting!) (3) and (4), we conclude $|c - c'| < 2\varepsilon$!
Since ε can be taken arbitrarily small, we must have c = c' (uniqueness).
From now on, *the* constant c in K_f will be denoted by m(f) : it is the only constant function which can be approximated arbitrarily close with convex combinations of left (or right) translates of f. The following properties are obvious

$$m(1) = 1 \qquad (K_f = \{1\} \text{ if } f = 1),$$
$$m(f) \geqslant 0 \text{ if } f \geqslant 0,$$
$$m(af) = a\, m(f) \text{ if } a \text{ is any complex number } (K_{af} = a K_f),$$
$$m({}_s f) = m(f) = m(f_s) \quad (\text{by uniqueness}).$$

Our proof will be complete if we show that m is *additive* (hence *linear*).
Let us take $f, g \in C(G)$ and start with (1) above with c = m(f). Put moreover

$$h(x) = \sum a_i g(s_i x) .$$

Since $h \in C_g$, we certainly have $C_h \subset C_g$ whence $K_h \subset K_g$. But the set K_g contains only one constant : m(h) = m(g). We can write

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$$\left| m(h) - \sum b_j h(t_j x) \right| < \varepsilon$$

for finitely many suitable $t_j \in G$, $b_j > 0$ and $\sum b_j = 1$. Using the definition of h and m(h) = m(g), we find

$$\left| m(g) - \sum_{i,j} a_i b_j g(s_i t_j x) \right| < \varepsilon \qquad (5).$$

But multiplying (1) by b_j, replacing x by $t_j x$ in it and summing over j we find

$$\left| m(f) - \sum_{i,j} a_i b_j f(s_i t_j x) \right| < \varepsilon \qquad (6).$$

Adding (5) and (6), we find

$$\left| m(f) + m(g) - \sum_{i,j} a_i b_j (f+g)(s_i t_j x) \right| < 2\varepsilon \quad .$$

Hence the constant m(f) + m(g) is in K_{f+g} : but the only constant of this compact convex set is m(f + g) . q.e.d.

This measure m on G is the (normalized) *Haar measure* of the compact group G. Instead of m(f), we shall often write

$$\int_G f(x)\, dm(x) \quad \text{or even, more simply} \quad \int f(x)\, dx \quad .$$

This notation will also be used for the (regular Borel) extended measure and integrable functions $f \in L^1(G) = L^1(G,m)$. It is countably additive. For example, the invariance under translations implies that points of G have same measure, and since m(G) = 1, we infer

$$m(\{e\}) > 0 \implies G \text{ finite group} \quad .$$

In this case, the (normalized) Haar measure of G is simply given by

$$m(\{e\}) = 1/n \quad (n = \text{Card}\, G) \ ,$$

$$m(f) = \frac{1}{n} \sum_{x \in G} f(x) \qquad (f \text{ in the } \textit{group algebra} \text{ of } G) \ .$$

In the opposite case, m({e}) = 0, all points have measure 0 and G cannot be countable (by countable additivity of m, m(N) = 0 for all countable sets in G : countable sets of G are negligible).

The measure of a subset of G is (by definition) the measure (or integral) of the characteristic function of this set. For measurable sets (or more simply Borel sets), m(A) is at the same time the supremum of the measures m(K) of the compact subsets of A, or the infimum of the measures m(U) of the open neighbourhoods U of A.

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EXERCISES

1. Let G be a compact group and m its Haar measure. Show that $m : C_{\mathbb{R}}(G) \longrightarrow \mathbb{R}$ is continuous ($C_{\mathbb{R}}(G)$ is the Banach space with respect to the sup norm). Hint : use $-\|f\| \leqslant f(x) \leqslant \|f\|$ $(f \in C_{\mathbb{R}}(G),\ x \in G)$.

2. Take for G the Cantor group. What is the measure of the subgroup defined by $a_o = 0$? Same question for the subgroups defined by $a_i = 0$ for $i \leqslant n$. Deduce from the preceding observations an expression for the Haar integral of a locally constant function. This gives a good "feeling" for the Haar integral of any continuous function (any such function is uniformly continuous hence uniformly approximable by locally constant functions and ex.1 can be applied).

Let now $I = [0,1]$ be the unit interval and consider G as embedded (topologically and metrically) in I. Show that G is a Lebesgue negligible set in I. The measure m on I (with support G) is not absolutely continuous with respect to Lebesgue measure (m is singular with respect to Lebesgue measure). The function f on I defined by $f(x) = m([0,x])$ is increasing and continuous. It is continuously differentiable in $I - G$ (remember that G is negligible) with $f'(x) = 0$ for $x \notin G$. However, f is not constant in I ! (One should recall that if f is differentiable outside a countable subset A of I with $f'(x) = 0$ for all $x \notin A$, then f is constant. One cannot replace the assumption A countable by A negligible.)

3. Let p be any prime number and consider the Haar measure m of the additive (compact) group $\mathbb{Z}_p$. What is the measure of the subgroups $p^n\mathbb{Z}_p$?

Show that the restriction of m to $\mathbb{Z}_p^{\times}$ is (proportional to) a Haar measure of this multiplicative group. Hint : observe that $1 + p\mathbb{Z}_p$ is a subgroup of index $p-1$ in the multiplicative group $\mathbb{Z}_p^{\times}$. Observe then that $1 + p^n\mathbb{Z}_p$ is a subgroup of index p^{n-1} of the multiplicative group $1 + p\mathbb{Z}_p$ (similarity with : $p^n\mathbb{Z}_p$ is a subgroup of index p^{n-1} in the additive group $p\mathbb{Z}_p$, and a subgroup of index p^n in $\mathbb{Z}_p$).

More precisely, show that the restriction of $p/(p-1)\cdot m$ to $\mathbb{Z}_p^{\times}$ is the normalized Haar measure of this multiplicative group.