1 REAL NUMBERS

1.1 Set notation

A *set* is a collection of objects which are called its *elements*. If x is an element of the set S, we say that x *belongs* to S and write

$$x \in S.$$

If y does not belong to S, we write $y \notin S$.

The simplest way of specifying a set is by listing its elements. We use the notation

$$A = \{\tfrac{1}{2}, 1, \sqrt{2}, e, \pi\}$$

to denote the set whose elements are the real numbers $\frac{1}{2}, 1, \sqrt{2}, e$ and π. Similarly

$$B = \{\text{Romeo, Juliet}\}$$

denotes the set whose elements are Romeo and Juliet.

This notation is, of course, no use in specifying a set which has an infinite number of elements. Such sets may be specified by naming the property which distinguishes elements of the set from objects which are not in the set. For example, the notation

$$C = \{x : x > 0\}$$

(which should be read 'the set of all x such that $x > 0$') denotes the set of all positive real numbers. Similarly

$$D = \{y : y \text{ loves Romeo}\}$$

denotes the set of all people who love Romeo.

It is convenient to have a notation for the *empty* set $\emptyset$. This is the set which has *no* elements. For example, if x denotes a variable which ranges over the set of all real numbers, then

$$\{x : x^2 + 1 = 0\} = \emptyset.$$

This is because there are no real numbers x such that $x^2 = -1$.

If S and T are two sets, we say that S is a *subset* of T and write

$$S \subset T$$

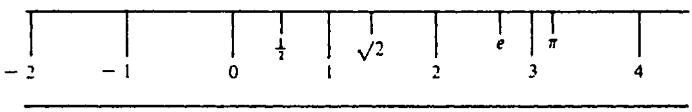
if every element of S is also an element of T.

As an example, consider the sets $P = \{1, 2, 3, 4\}$ and $Q = \{2, 4\}$. Then $Q \subset P$. Note that this is *not* the same thing as writing $Q \in P$, which means that Q is an element of P. The elements of P are simply 1, 2, 3 and 4. But Q is not one of these.

The sets A, B, C and D given above also provide some examples. We have $A \subset C$ and (presumably) $B \subset D$.

1.2 The set of real numbers

It will be adequate for this book to think of the real numbers as being points along a straight line which extends indefinitely in both directions. The line may then be regarded as an ideal ruler with which we may measure the lengths of line segments in Euclidean geometry.



The set of all real numbers will be denoted by $\mathbb{R}$. The table below distinguishes three important subsets of $\mathbb{R}$.

Subset	Notation	Elements
Natural numbers (or whole numbers)	$\mathbb{N}$	$1, 2, 3, 4, 5, \ldots$
Integers	$\mathbb{Z}$	$\ldots -2, -1, 0, 1, 2, 3, \ldots$
Rational numbers (or fractions)	$\mathbb{Q}$	$0, 1, 2, -1, \frac{1}{2}, \frac{3}{4}, \frac{5}{3}, -\frac{1}{2}, -\frac{3}{7}, \ldots$

Not all real numbers are rational. Some examples of irrational numbers are $\sqrt{2}$, e and π.

While we do not go back to first principles in this book, the treatment will be rigorous in so far as it goes. It is therefore important to be clear, at every stage, about what our assumptions are. We shall then know what has to be proved and what may be taken for granted. Our most vital assumptions are concerned with the properties of the real number system. The rest of this chapter and the following two chapters are consequently devoted to a description of the properties of the real number system which we propose to assume and to some of their immediate consequences. A very much more systematic account of these assumptions is given in the author's book *Logic, Sets and Numbers* (see pp. 44–77).

1.3 Arithmetic

The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.

The rules of arithmetic, of course, include the proviso that division by zero is not allowed. Thus, for example, the expression

$$\frac{2}{0}$$

makes no sense at all. In particular, it is *not* true that

$$\frac{2}{0} = \infty.$$

We shall have a great deal of use for the symbol ∞, but it must clearly be understood that ∞ does *not* represent a real number. Nor can it be treated as such except in very special circumstances.

1.4 Inequalities

The next assumptions concern inequalities between real numbers and their manipulation.

We assume that, given any two real numbers a and b, there are three mutually exclusive possibilities:

(i) $a > b$ (a is greater than b)

(ii) $a = b$ (a equals b)

(iii) $a < b$ (a is less than b).

Observe that $a < b$ means the same thing as $b > a$. We have, for example, the following inequalities.

$$1 > 0;\ 3 > 2;\ 2 < 3;\ -1 < 0;\ -3 < -2.$$

There is often some confusion about the statements

(iv) $a \geqslant b$ (a is greater than *or* equal to b)

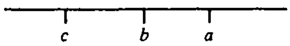
(v) $a \leqslant b$ (a is less than *or* equal to b).

To clear up this confusion, we note that the following are all true statements.

$$1 \geqslant 0;\ 3 \geqslant 2;\ 1 \geqslant 1;\ 2 \leqslant 3;\ -1 \leqslant 0;\ -3 \leqslant -3.$$

We assume four basic rules for the manipulation of inequalities. From these the other rules may be deduced.

(I) If $a > b$ and $b > c$, then $a > c$.



(II) If $a > b$ and c is any real number, then

$$a + c > b + c.$$

(III) If $a > b$ and $c > 0$, then $ac > bc$ (i.e. inequalities can be multiplied through by a *positive* factor).

(IV) If $a > b$ and $c < 0$, then $ac < bc$ (i.e. multiplication by a *negative* factor reverses the inequality).

1.5 *Example* If $a > 0$, prove that $a^{-1} > 0$.

Proof We argue by contradiction. Suppose that $a > 0$ but that $a^{-1} \leqslant 0$. It cannot be true that $a^{-1} = 0$ (since then $0 = 0.a = 1$). Hence

$$a^{-1} < 0.$$

By rule III we can multiply this inequality through by a (since $a > 0$). Hence

$$1 = a^{-1}.a < 0.a = 0.$$

But $1 < 0$ is a contradiction. Therefore the assumption $a^{-1} \leqslant 0$ was false. Hence $a^{-1} > 0$.

1.6 *Example* If x and y are positive, then $x < y$ if and only if $x^2 < y^2$.

Proof We have to show *two* things. First, that $x < y$ implies $x^2 < y^2$, and secondly, that $x^2 < y^2$ implies $x < y$.

(i) We begin by assuming that $x < y$ and try to deduce that $x^2 < y^2$. Multiply the inequality $x < y$ through by $x > 0$ (rule III). We obtain

$$x^2 < xy.$$

Similarly

$$xy < y^2.$$

But now $x^2 < y^2$ follows from rule I.

(ii) We now assume that $x^2 < y^2$ and try and deduce that $x < y$. Adding $-x^2$ to both sides of $x^2 < y^2$ (rule II), we obtain

$$y^2 - x^2 > 0$$

i.e. $$(y - x)(y + x) > 0. \qquad (1)$$

Since $x + y > 0$, $(x + y)^{-1} > 0$ (example 1.5). We can therefore multiply through inequality (1) by $(x + y)^{-1}$ to obtain

$$y - x > 0$$

i.e. $\quad x < y.$

(Alternatively, we could prove (ii) as follows. Assume that $x^2 < y^2$ but that $x \geqslant y$. From $x \geqslant y$ it follows (as in (i)) that $x^2 \geqslant y^2$, which is a contradiction.)

1.7 *Example* Suppose that, for *any* $\epsilon > 0, a < b + \epsilon$. Then $a \leqslant b$.

Proof Assume that $a > b$. Then $a - b > 0$. But, for *any* $\epsilon > 0$, $a < b + \epsilon$. Hence $a < b + \epsilon$ in the particular case when $\epsilon = a - b$. Thus

$$a < b + (a - b)$$

and so $\quad a < a.$

This is a contradiction. Hence our assumption $a > b$ must be false. Therefore $a \leqslant b$.
(Note: The symbol ϵ in this example is the Greek letter *epsilon*. It should be carefully distinguished from the 'belongs to' symbol $\in$ and also from the symbol ξ which is the Greek letter *xi*.)

1.8 *Exercise*

(1) If x is any real number, prove that $x^2 \geqslant 0$. If $0 < a < 1$ and $b > 1$, prove that

(i) $0 < a^2 < a < 1$ (ii) $b^2 > b > 1$.

(2) If $b > 0$ and $B > 0$ and

$$\frac{a}{b} < \frac{A}{B},$$

prove that $aB < bA$. Deduce that

$$\frac{a}{b} < \frac{a + A}{b + B} < \frac{A}{B}.$$

(3) If $a > b$ and $c > d$, prove that $a + c > b + d$ (i.e. inequalities can be added). If, also, $b > 0$ and $d > 0$, prove that $ac > bd$ (i.e. inequalities between *positive* numbers can be multiplied).

(4) Show that each of the following inequalities may fail to hold even though $a > b$ and $c > d$.

(i) $a - c > b - d$

(ii) $\frac{a}{c} > \frac{b}{d}$

(iii) $ac > bd$.

What happens if we impose the extra condition that $b > 0$ and $d > 0$?

(5) Suppose that, for *any* $\epsilon > 0$, $a - \epsilon < b < a + \epsilon$. Prove that $a = b$.

(6) Suppose that $a < b$. Show that there exists a real number x satisfying $a < x < b$.

1.9 Roots

Let n be a natural number. The reader will be familiar with the notation $y = x^n$. For example, $x^2 = x.x$ and $x^3 = x.x.x$.

Our next assumption about the real number system is the following. Given any $y \geqslant 0$ there is exactly one value of $x \geqslant 0$ such that

$$y = x^n.$$

(Later on we shall see how this property may be deduced from the theory of continuous functions.)

If $y \geqslant 0$, the value of $x \geqslant 0$ which satisfies the equation $y = x^n$ is called the nth *root* of y and is denoted by

$$x = y^{1/n}.$$

When $n = 2$, we also use the notation $\sqrt{y} = y^{1/2}$. Note that, with this convention, it is always true that $\sqrt{y} \geqslant 0$. If $y > 0$, there are, of course, *two* numbers whose square is y. The positive one is $\sqrt{y}$ and the negative one is $-\sqrt{y}$. The notation $\pm\sqrt{y}$ means '$\sqrt{y}$ *or* $-\sqrt{y}$'.

If $r = m/n$ is a positive rational number and $y \geqslant 0$, we define

$$y^r = (y^m)^{1/n}.$$

If r is a negative rational, then $-r$ is a positive rational and hence y^{-r} is defined. If $y > 0$ we can therefore define y^r by

$$y^r = \frac{1}{y^{-r}}.$$

We also write $y^0 = 1$. With these conventions it follows that, if $y > 0$, then y^r is defined for all rational numbers r. (The definition of y^x when x is an irrational real number must wait until a later chapter.)

1.10 Quadratic equations

If $y > 0$, the equation $x^2 = y$ has two solutions. We denote the *positive* solution by $\sqrt{y}$. The *negative* solution is therefore $-\sqrt{y}$. We note again that there is no ambiguity about these symbols and that $\pm\sqrt{y}$ simply means '$\sqrt{y}$ or $-\sqrt{y}$'.

The general quadratic equation has the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. Multiply through by $4a$. We obtain

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax + b)^2 - b^2 + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac.$$

It follows that the quadratic equation has no real solutions if $b^2 - 4ac < 0$, one real solution if $b^2 - 4ac = 0$ and two real solutions if $b^2 - 4ac > 0$. If $b^2 - 4ac \geqslant 0$,

$$2ax + b = \pm\sqrt{(b^2 - 4ac)}$$

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

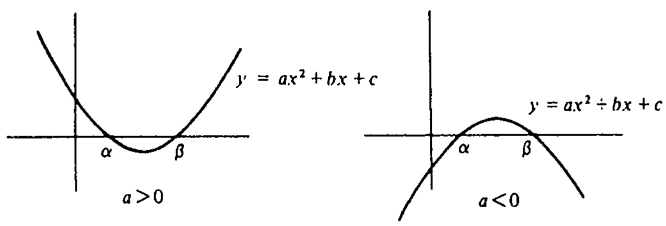
The roots of the equation $ax^2 + bx + c = 0$ are therefore

$$\alpha = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad \beta = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}.$$

It is a simple matter to check that, for all values of x,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

With the help of this formula, we can sketch the graph of the equation $y = ax^2 + bx + c$.



1.11 *Example* A nice application of the work on quadratic equations described above is the proof of the important *Cauchy–Schwarz inequality*. This asserts that, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, then

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2).$$

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K. G. Binmore
Excerpt
More information

Proof For any x,

$$0 \leqslant (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2$$
$$= (a_1^2 + \ldots + a_n^2)x^2 + 2(a_1 b_1 + \ldots + a_n b_n)x + (b_1^2 + \ldots + b_n^2)$$
$$= Ax^2 + 2Bx + C.$$

Since $y = Ax^2 + 2Bx + C \geqslant 0$ for *all* values of x, it follows that the equation $Ax^2 + 2Bx + C = 0$ cannot have two (distinct) roots. Hence

$$(2B)^2 - 4AC \leqslant 0$$

i.e. $\quad B^2 \leqslant AC$

which is what we had to prove.

1.12 *Exercise*

(1) Suppose that n is an *even* natural number. Prove that the equation $x^n = y$ has no solutions if $y < 0$, one solution if $y = 0$ and two solutions if $y > 0$.

Suppose that n is an *odd* natural number. Prove that the equation $x^n = y$ always has one and only one solution.

Draw graphs of $y = x^2$ and $y = x^3$ to illustrate these results.

(2) Simplify the following expressions:

(i) $8^{2/3}$ (ii) $27^{-4/3}$ (iii) $32^{6/5}$.

(3) If $y > 0$, $z > 0$ and r and s are any rational numbers, prove the following:

(i) $y^{r+s} = y^r y^s$ (ii) $y^{rs} = (y^r)^s$ (iii) $(yz)^r = y^r z^r$.

(4) Suppose that $a > 0$ and that α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ (in which $b^2 - 4ac > 0$). Prove that $y = ax^2 + bx + c$ is negative when $\alpha < x < \beta$ and positive when $x < \alpha$ or $x > \beta$. Show also (without the use of calculus) that $y = ax^2 + bx + c$ achieves a minimum value of $c - b^2/4a$ when $x = -b/2a$.

(5) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Their arithmetic mean A_n and harmonic mean H_n are defined by

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad H_n^{-1} = \frac{1}{n}\left\{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right\}.$$

Deduce from the Cauchy–Schwarz inequality that $H_n \leqslant A_n$.

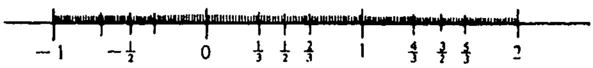
(6) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any real numbers. Prove Minkowski's inequality, i.e.

$$\left\{\sum_{k=1}^{n} (a_k + b_k)^2\right\}^{1/2} \leqslant \left\{\sum_{k=1}^{n} a_k^2\right\}^{1/2} + \left\{\sum_{k=1}^{n} b_k^2\right\}^{1/2}.$$

For the case $n = 2$ (or $n = 3$) this inequality amounts to the assertion that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Explain this.

1.13 Irrational numbers

In § 1.2 we mentioned the existence of irrational real numbers. That such numbers exist is by no means obvious. For example, one may imagine the process of marking all the rational numbers on a straight line. First one would mark the integers. Then one would move on to the multiples of $\frac{1}{2}$ and then to the multiples of $\frac{1}{3}$ and so on. Assuming that this program could ever be completed, one might very well be forgiven for supposing that there would be no room left for any more points on the line.



But our assumption about the existence of nth roots renders this view untenable. This assumption requires us to accept the existence of a positive real number x (namely $\sqrt{2}$) which satisfies $x^2 = 2$. If x were a rational number it would be expressible in the form

$$x = \frac{m}{n}$$

where m and n are natural numbers with no common divisor (other than 1). It follows that

$$m^2 = 2n^2$$

and so m^2 is even. This implies that m is even. (If m were odd, we should have $m = 2k + 1$. But then $m^2 = 4k^2 + 4k + 1$ which is odd.) We may therefore write $m = 2k$. Hence

$$4k^2 = 2n^2$$
$$n^2 = 2k^2.$$

Thus n is even. We have therefore shown that both m and n are divisible by 2. This is a contradiction and it follows that x cannot be rational, i.e. $\sqrt{2}$ must be an irrational real number.

Of course, $\sqrt{2}$ is not the only irrational number and the ability to extract nth roots allows us to construct many others. But it should not be supposed that all irrational numbers can be obtained in this way. It is not even true that every irrational number is a root of an equation of the form

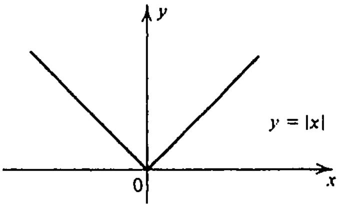
$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

in which the coefficients $a_0, a_1, \ldots, a_n$ are rational numbers. Real numbers which are not the roots of such an equation are called transcendental, notable examples being e and π. This fascinating topic, however, lies outside the scope of this book.

1.14 Modulus

Suppose that x is a real number. Its *modulus* (or *absolute value*) $|x|$ is defined by

$$|x| = \begin{cases} x & (x \geqslant 0) \\ -x & (x < 0). \end{cases}$$



Thus $|3| = 3$, $|-6| = 6$ and $|0| = 0$. Obviously $|x| \geqslant 0$ for all values of x.

It is sometimes useful to note that $|x| = \sqrt{x^2}$.

1.15 *Theorem* For any real number x,

$$-|x| \leqslant x \leqslant |x|.$$

Proof Either $x \geqslant 0$ or $x < 0$. In the first case, $-|x| \leqslant 0 \leqslant x = |x|$. In the second case, $-|x| = x < 0 < |x|$.

1.16 *Theorem* For any real numbers a and b

$$|ab| = |a|.|b|$$

Proof The most elegant proof is the following.

$$|ab| = \sqrt{((ab)^2)} = \sqrt{(a^2b^2)} = \sqrt{(a^2)}.\sqrt{(b^2)} = |a|.|b|.$$

The next theorem is of great importance and is usually called the *triangle inequality*. This nomenclature may be justified by observing that the theorem is the special case of Minkowski's inequality (exercise 1.12(6)) with $n = 1$. We give a separate proof.

1.17 *Theorem (triangle inequality)* For any real numbers a and b,

$$|a + b| \leqslant |a| + |b|.$$

Proof We have