1 EXAMPLES OF ISOLATED SINGULAR POINTS

Loosely speaking, an analytic germ (X,x) in $\mathbb{C}^N$ is called a complete intersection if the minimal number of equations by which it can be defined equals its codimension in $\mathbb{C}^N$. Although such germs will be our principal object of study, we must realize that quite often analytic germs are not given to us as the common zero set of a specific set of equations. In such cases it is unreasonable to expect these germs to be complete intersections. We illustrate this by describing several constructions of singular germs, most of which fail to yield complete intersections in general. Some of the germs which happen to be complete intersections, will reappear when we make a beginning of the classification in chapter 7. Another goal of this chapter is to make the reader acquainted with several interesting examples to which the theory we are going to develop may be applied. We do not always provide full proofs of the properties attributed to these singularities. The reader shouldn't feel uneasy about this, for in such cases we will not make any use of them.

1.A *Hypersurface singularities*

(1.1) Let X be an analytic set in an open $U \subset \mathbb{C}^{n+1}$ and let $x \in X$. The ideal $I_{X,x}$ of holomorphic functions at x vanishing on X is principal and nonzero if and only if each irreducible component of the germ (X,x) is

Cambridge University Press
978-0-521-28674-9 - Isolated Singular Points on Complete Intersections
E. J. N. Looijenga
Excerpt
More information

of dimension n (see for instance Whitney (1972), Ch. 2, Thm's 10 C,D). We then say that (X,x) is a *hypersurface germ*. If $f \in m_{\mathbb{C}^{n+1},x}$ generates $I_{X,x}$, then the fact that I_X is a coherent O_U-module implies that there is an open neighbourhood U' of x in U such that f converges on U' and $I_X|U' = fO_{U'}$.

(1.2) *Proposition.* In this situation, the following are equivalent:

(i) There is a neighbourhood U'' of x in U such that $(X \cap U'')-\{x\}$ is non-singular.

(ii) $(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x} \supset m^k_{U,x}$ for some k.

(iii) $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x} \supset m^k_{U,x}$ for some k.

(iv) $\dim_{\mathbb{C}} O_{U,x}/(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x} < \infty$.

(v) $\dim_{\mathbb{C}} O_{U,x}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x} < \infty$.

Proof. (i) ⇒ (ii). If y is a nonsingular point of $X \cap U'$, then $\frac{\partial f}{\partial z_\nu}(y) \neq 0$ for some ν, for f generates $I_{X,y}$. So the common zero set of $f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$ is contained in the singular locus $C_{X \cap U'}$ of $X \cap U'$. Since $C_{X \cap U''} \subset \{x\}$ (by assumption), the radical of $(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x}$ is either $O_{U,x}$ or $m_{U,x}$ by the local analytic Nullstellensatz. This clearly implies (ii).

(ii) ⇒ (iii). Let $Y \subset U'$ denote the common zero set of $\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$. It will be enough to show that $(Y,x) \subset (X,x)$ because the Nullstellensatz will then imply that $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x}$ and $(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})O_{U,x}$ have the same radical. If $(Y,x) \not\subset (X,x)$, then we can find a germ of an analytic curve $s : (\mathbb{C},0) \to (Y,x)$ with $s(t) \notin X$ for $t \neq 0$. But

$$\frac{d}{dt}(f \circ s) = \Sigma^n_{\nu=0} (\frac{\partial f}{\partial z_\nu} \circ s) \frac{ds_\nu}{dt} = 0$$

and so $f \circ s$ is constant equal to $f \circ s(0) = 0$. This contradicts our assumption that $s(t) \notin X$ for $t \neq 0$.

(iii) ⇒ (iv), (iv) ⇒ (v) and (ii) ⇒ (i) are easy, while (v) ⇒ (ii) will follow from the lemma below applied to $M = \mathcal{O}_{U,x}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\mathcal{O}_{U,x}$.

(1.3) *Lemma*. If M is an $\mathcal{O}_{U,x}$-module of finite $\mathbb{C}$-dimension d, then $m^d_{U,x}$ annihilates M.

Proof. Put $d_k := \dim_{\mathbb{C}} M/m^k_{U,x}M$, $k = 0,1,2,\ldots$. Then $0 = d_0 \leq d_1 \leq \ldots$ $\ldots \leq d_k \leq \ldots \leq d$. So for some $k \leq d$, we have $d_k = d_{k+1}$. This means that $m^k_{U,x}M = m^{k+1}_{U,x}M$. Since $m^k_{U,x}M \subset M$ is of finite $\mathbb{C}$-dimension, $m^k_{U,x}M$ is a noetherian $\mathcal{O}_{U,x}$-module so that Nakayama's lemma applies: it follows that $m^k_{U,x}M = 0$.

(1.4) If one of the (equivalent) conditions of (1.2) is satisfied we say that (X,x) is an *isolated hypersurface singularity*. The dimension occurring in (1.2)-iv is usually called the *Milnor number* of X at x and denoted $\mu(X,x)$. Milnor (1968) originally defined this number in a topological manner but we will see in chapter 5 that the two definitions agree. The dimension in (1.2)-v will be interpreted in chapter 6 when we investigate the deformation theory of (X,x). We follow Greuel and call it the *Tjurina number* of (X,x), denoted $\tau(X,x)$. Clearly $\tau(X,x) \leq \mu(X,x)$ and we have equality if and only if $f \in (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\mathcal{O}_{U,x}$. This is for instance the case if f is *weighted homogeneous*. This means that there exist positive integers $d_0,\ldots,d_n,N$ such that if we give z_ν degree d_ν, f is a homogeneous polynomial of degree N, in other words f is of the form

$$f(z) = \Sigma_{i_0 d_0 + \ldots + i_n d_n = N} \quad a_{i_0 \ldots i_n} \quad z_0^{i_0} \ldots z_n^{i_n}.$$

(We take $x = 0$.) Then it is easily checked that

$$f(z) = \Sigma_{\nu=0}^{n} \frac{d_\nu}{N} z_\nu \frac{\partial f}{\partial z_\nu} .$$

The condition that $f \in (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\mathcal{O}_{U,x}$ is coordinate invariant and hence also satisfied for an f which is weighted homogeneous with respect to some coordinate system at $(\mathbb{C}^{n+1},x)$. According to Saito (1971) there is a converse to this: if f defines an isolated hypersurface singularity and $f \in (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})\mathcal{O}_{U,x}$, then f is weighted homogeneous with respect to some coordinate system.

1.B *Complete intersections*

(1.5) Let x be a point of an analytic set X, defined in an open $U \subset \mathbb{C}^N$, and let n denote the dimension of X at x. Then near x, X cannot be defined as the common zero set of fewer than N-n holomorphic functions at x. If we can do it with N-n such functions, then we say that X is a *geometric complete intersection* at x. This is a nontrivial condition: for instance the union of the (z_1,z_2)-plane and the (z_3,z_4)-plane in $\mathbb{C}^4$ is not a geometric complete intersection at the origin, see for instance Gunning (1974), p. 159. Likewise if $I \subset m_{\mathbb{C}^N,x}$ is an ideal which defines a germ of dim n, then we say that I defines a *complete intersection* at x if I admits N-n generators $f_1,\ldots,f_{N-n}$. The following result characterizes such (N-n)-tuples algebraically.

(1.6) Let $f_1,\ldots,f_{N-n} \in m_{\mathbb{C}^N,x}$ generate an ideal I in $\mathcal{O}_{\mathbb{C}^N,x}$. Then I defines a complete intersection of dim n if and only if $f_1,\ldots,f_{N-n}$ is an $\mathcal{O}_{\mathbb{C}^N,x}$*-sequence*, i.e. f_j is not a zero-divisor of $\mathcal{O}_{\mathbb{C}^N,x}/(f_1,\ldots,f_{j-1})\mathcal{O}_{\mathbb{C}^N,x}$ for $j=1,\ldots,N-n$. If either condition is fulfilled, $\mathcal{O}_{\mathbb{C}^N,x}/I$ is a Cohen-

Macaulay ring of dim n. In particular, dim $\mathcal{O}_{\mathbb{C}^N,x}/P = n$ for any associated prime ideal P of $\mathcal{O}_{\mathbb{C}^N,x}/I$.

For a proof, see for instance Matsumura (1980), Th. 30.

The property that an ideal $I \subset m_{\mathbb{C}^N,x}$ defines a complete intersection at x only depends on the $\mathbb{C}$-algebra $\mathcal{O}_{\mathbb{C}^N,x}/I$, as will follow from (1.8) below. Let us start with proving an intermediate result, which we shall use at other places as well.

(1.7) *Lemma*. Let $I \subset \mathcal{O}_{\mathbb{C}^N,0}$ and $J \subset \mathcal{O}_{\mathbb{C}^N,0}$ be ideals and assume we are given an isomorphism of $\mathbb{C}$-algebras $\phi^* : \mathcal{O}_{\mathbb{C}^N,0}/J \to \mathcal{O}_{\mathbb{C}^N,0}/I$. Then there exists an analytic automorphism $\Phi : (\mathbb{C}^N,0)\circlearrowleft$ with $\Phi^*(J) = I$ such that Φ^* induces ϕ^*.

Proof. We assume that I and J are both distinct from $\mathcal{O}_{\mathbb{C}^N,0}$. Let r denote the dimension of the $\mathbb{C}$-vector space $(I+m^2)/m^2$ (where $m = m_{\mathbb{C}^N,0}$). Choose $z'_1,\ldots,z'_r \in I$ such that their images in $(I+m^2)/m^2$ give a basis and extend this set to a coordinate system $z'_1,\ldots,z'_N$ for $(\mathbb{C}^N,0)$. In the exact sequence

$$0 \to (I+m^2)/m^2 \to m/m^2 \to m/(I+m^2) \to 0$$

the term $m/(I+m^2)$ is naturally isomorphic to m_I/m_I^2 where $m_I \subset \mathcal{O}_{\mathbb{C}^N,0}/I$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^N,0}/I$. Since $\mathcal{O}_{\mathbb{C}^N,0}/I$ and $\mathcal{O}_{\mathbb{C}^N,0}/J$ are isomorphic, it follows that $m/(I+m^2)$ and $m/(J+m^2)$ have the same $\mathbb{C}$-dimension. Hence $\dim_{\mathbb{C}}(J+m^2)/m^2 = r$. Repeating the above construction with J yields a coordinate system $z''_1,\ldots,z''_N$ for $(\mathbb{C}^N,0)$ with $z''_1,\ldots,z''_r \in J$. Choose $\Phi_\nu \in m_{\mathbb{C}^N,0}$ $(\nu = r+1,\ldots,N)$ such that its reduction mod I is just the image of $z''_\nu + J$ under ϕ^*. Define $\Phi : (\mathbb{C}^N,0)\circlearrowleft$ by $\Phi^*(z''_\nu) = z'_\nu$ if $\nu \le r$ and $\Phi^*(z''_\nu) = \Phi_\nu$ if $\nu > r$. Then Φ induces a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & (J+m^2)/m^2 & \to & m/m^2 & \to & m_J/(m_J)^2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (I+m^2)/m^2 & \to & m/m^2 & \to & m_I/(m_I)^2 & \to & 0 \end{array}$$

The vertical map on the left is an isomorphism by construction, whereas the one on the right is so because it is induced by ϕ^*. Hence the middle map is an isomorphism. As this represents the derivative of Φ in 0 it follows that Φ is an isomorphism. Clearly, $\Phi^*(J) \subset I$. Since Φ^* induces ϕ^*, we have in fact $\Phi^*(J) = I$.

(1.8) *Lemma*. Let $I \subset m_{\mathbb{C}^N,0}$ and $J \subset m_{\mathbb{C}^M,0}$ be ideals such that $A := \mathcal{O}_{\mathbb{C}^N,0}/I$ and $B := \mathcal{O}_{\mathbb{C}^M,0}/J$ are isomorphic $\mathbb{C}$-algebras. If $d(I)$ respectively $d(J)$ denotes the minimal number of generators of I respectively J, then $N-d(I) = M-d(J)$.

Proof. Put $r := \dim_{\mathbb{C}}(I+m^2)/m^2$. If I admits the generators $g_1,\ldots,g_{d(I)} \in m_{\mathbb{C}^N,0}$ then by making a suitable coordinate transformation of $(\mathbb{C}^N,0)$, we may assume that $g_i = z_{N-r+i}$ for $i \le r$ and $g_i \in m^2_{\mathbb{C}^N,0}$ for $i > r$. Let $\pi : \mathcal{O}_{\mathbb{C}^N,0} \to \mathcal{O}_{\mathbb{C}^{N-r},0}$ denote the obvious projection and set $I' = \pi(I)$. Then $\pi^{-1}(I') = I$, so that π induces an isomorphism of $\mathcal{O}_{\mathbb{C}^N,0}/I$ onto $\mathcal{O}_{\mathbb{C}^{N-r},0}/I'$. As I' is generated by $\pi(g_{r+1}),\ldots,\pi(g_{d(I)})$, it follows that $d(I') \le d(I)-r$. On the other hand, if $g',\ldots,g'_{d(I')}$ generate I', then lifts of these in I toghether with $z_{N-r+1},\ldots,z_N$ generate I, so that $d(I) \le d(I')+r$, also. It follows that $d(I') = d(I)-r$, where I' has now the virtue of being contained in $m^2_{\mathbb{C}^{N-r},0}$. So without loss of generality we may assume that I and J are in the squares of the maximal ideals. We claim that then $N = M$. This simply follows from the fact that $m_{\mathbb{C}^N,0}/m^2_{\mathbb{C}^N,0}$ maps isomorphically onto $m_{\mathbb{C}^N,0}/(I+m^2_{\mathbb{C}^N,0}) \cong m_A/m_A^2$, where m_A denotes the maximal ideal of A, so that N is invariantly characterized as $\dim_{\mathbb{C}} m_A/m_A^2$. Similarly, $M = \dim_{\mathbb{C}} m_B/m_B^2$ and so $N = M$. The lemma now follows from (1.7).

Cambridge University Press
978-0-521-28674-9 - Isolated Singular Points on Complete Intersections
E. J. N. Looijenga
Excerpt
More information

(1.9) The preceding argument shows that for any local analytic algebra A (i.e. a $\mathbb{C}$-algebra isomorphic to one of the form $\mathcal{O}_{\mathbb{C}^N,0}/I$) the minimal N such that A is isomorphic to some quotient of $\mathcal{O}_{\mathbb{C}^N,0}$ is given by $\dim_{\mathbb{C}} m_A/m_A^2$. This number is called the *embedding dimension* of A. The difference $\dim m_A/m_A^2 - \dim A$ is called the *embedding codimension* of A. We say that a local analytic $\mathbb{C}$-algebra A is a *(local) complete intersection algebra* if for some surjection $\pi : \mathcal{O}_{\mathbb{C}^N,x} \to A$ of local $\mathbb{C}$-algebras, $I := \mathrm{Ker}(\pi)$ defines a complete intersection (X,x) in the previous sense (by the preceding lemma, this is then so for *any* such π). The case that concerns us most is when X has an isolated singular point or is regular in x. This means that if $f_1,\ldots,f_{N-n}$ is a set of generators of I (with $n = \dim A$) then there is an open neighbourhood V of x in $\mathbb{C}^N$ on which $f_1,\ldots,f_{N-n}$ converge and for all $y \neq x$ in the common zero set X of $f_1,\ldots,f_{N-n}$, $df_1(y),\ldots,df_{N-n}(y)$ are linearly independent. This is also equivalent to the condition that the ideal in $\mathcal{O}_{\mathbb{C}^N,x}$ generated by $f_1,\ldots,f_{N-n}$ and the determinants of the $(N-n)\times(N-n)$ submatrices of $(\frac{\partial f_j}{\partial z_\nu})$ contains a power of $m_{\mathbb{C}^N,x}$. We then say that (X,x) endowed with its local $\mathbb{C}$-algebra $\mathcal{O}_{\mathbb{C}^N,x}/I$ ($\cong$ A) is an *isolated complete intersection singularity* (so this includes the case that $\mathcal{O}_{\mathbb{C}^N,x}/I$ is regular). Henceforth we shall abbreviate this as *icis*. We will often use this in expressions like: $(f_1,\ldots,f_{N-n}) : (\mathbb{C}^N,x) \to (\mathbb{C}^{N-n},0)$ (or $I \subset \mathcal{O}_{\mathbb{C}^N,x}$, or A) defines an icis.

(1.10) *Proposition.* If $I \subset \mathcal{O}_{\mathbb{C}^N,x}$ defines an icis of dim $n > 0$, then I is its own radical.

Proof. Let $f_1,\ldots,f_{N-n} \in I$, V and X be as above. We want to show that the sheaf $F := I_X/(f_1,\ldots,f_{N-n})\mathcal{O}_V$ is trivial. This is clearly the case outside $\{x\}$. Since F is a coherent sheaf of $\mathcal{O}_V$-modules, its annihilator $\mathrm{Ann}(F) = \{f \in \mathcal{O}_V : fF = 0\}$ is also coherent. Since $\mathrm{Ann}(F_y) = \mathcal{O}_{V,y}$ for

$y \neq x$, we must have $\mathrm{Ann}(F_x) \supset m_{V,x}^{\ell}$ for some ℓ (by the Nullstellensatz). Either $\ell = 0$ (and hence $F_x = \{0\}$ as was to be shown) or $\ell > 0$ and then each element of F_x must be a zero-divisor. According to (1.6) this can only happen when $n = 0$.

Example 1. The pair of quadratic forms in $\mathbb{C}^N$ $(N \geq 2)$,

$$f_1(z_1,\dots,z_N) = z_1^2+\dots+z_N^2$$

$$f_2(z_1,\dots,z_N) = \lambda_1 z_1^2+\dots+\lambda_N z_N^2$$

defines an icis of dim N-2 at $0 \in \mathbb{C}^N$ if and only if the coefficients $\lambda_1,\dots,\lambda_N$ are all distinct. This illustrates a result due to Hamm (1969) which says that for a given $(N-n)\times N$ matrix $(n_{j\nu})$ with $n_{j\nu} \in \mathbb{N}$, the equations

$$f_j(z_1,\dots,z_N) = \Sigma_{\nu=1}^N \, a_{j\nu} z_\nu^{n_{j\nu}}$$

define an icis for almost any coefficient matrix $(a_{j\nu})$.

1.C *Quotient singularities*

Let G be a finite group of local analytic automorphisms of $\mathbb{C}^n$ at 0. Following Cartan (1957), this action can be linearized, that is, in terms of a (possibly) new coordinate system for $(\mathbb{C}^n,0)$, G will act linearly. Therefore it is no restriction to assume that G is a subgroup of $GL_n(\mathbb{C})$. Then G also acts on $\mathbb{C}[z_1,\dots,z_n]$ by $(g.\phi)(z) = \phi(g^{-1}(z))$. The G-invariant polynomials form a homogeneous subalgebra $\mathbb{C}[z_1,\dots,z_n]^G$ of $\mathbb{C}[z_1,\dots,z_n]$, which is finitely generated and normal, see (Bourbaki: AC V), §1, no. 9. Choose homogeneous generators $\phi_1,\dots,\phi_N$ of $\mathbb{C}[z_1,\dots,z_n]^G$ of positive degree and define a polynomial mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^N$ by $\Phi(z) = (\phi_1(z),\ldots,\phi_N(z))$. Then Φ is constant on the G-orbits and thus factors through the orbit space $G\backslash\mathbb{C}^n$ by a mapping $\Phi' : G\backslash\mathbb{C}^n \to \mathbb{C}^N$.

(1.11) *Proposition*. The mapping Φ' is a proper homeomorphism of $G\backslash\mathbb{C}^n$ onto a normal algebraic subvariety of $\mathbb{C}^N$ whose algebra of regular functions corresponds under Φ to $\mathbb{C}[z_1,\ldots,z_n]^G$.

Proof. Let us first show that Φ' is injective. For this it suffices to check that given $z,z' \in \mathbb{C}^n$, $z' \notin G.z$, we have $\Phi(z) \neq \Phi(z')$. Choose a polynomial $\psi \in \mathbb{C}[z_1,\ldots,z_n]$ with $\psi(z') = 0$ and $\psi(g.z) \neq 0$ for all $g \in G$ and put $\psi_0 := \pi_{g\in G}(g.\psi)$. Then $\psi_0 \in \mathbb{C}[z_1,\ldots,z_n]^G = \mathbb{C}[\phi_1,\ldots,\phi_N]$. Since $\psi_0(z') = 0 \neq \psi_0(z)$, we must have $\phi_\nu(z) \neq \phi_\nu(z')$ for some ν.

Next we show that Φ is proper. Since Φ' is injective, we have $\Phi'^{-1}(0) = G.\{0\} = \{0\}$. Since the unit sphere of $\mathbb{C}^n$ is compact there exists an $\varepsilon > 0$ such that $\max\{|\phi_1(z)|,\ldots,|\phi_\nu(z)|\} \geq \varepsilon$ if $|z| = 1$. If d_ν is the degree of ϕ_ν, then it follows that

$$|\Phi(z)|^2 = |z|^{2d_1}|\phi_1(\tfrac{z}{|z|})|^2 + \ldots + |z|^{2d_N}|\phi_N(\tfrac{z}{|z|})|^2$$
$$\geq \varepsilon^2 \min.\{|z|^{2d_1},\ldots,|z|^{2d_N}\}.$$

This shows that the pre-image of a bounded set under Φ is bounded and hence that Φ is proper.

Since Φ' is a proper continuous injection, it is a homeomorphism onto its image. A basic result in elimination theory, e.g. Mumford (1976), (2.31), (2.33), implies that the image X of the proper algebraic mapping $\Phi = \mathbb{C}^n \to \mathbb{C}^N$ is algebraic in $\mathbb{C}^N$. Clearly, Φ^* maps the algebra of regular functions on X isomorphically onto $\mathbb{C}[z_1,\ldots,z_n]^G$. The latter is normal and hence so is X.

This result can be understood in a more conceptual way as saying that the orbit space $G\backslash\mathbb{C}^n$ is in a natural manner a normal affine algebraic variety whose algebra of regular functions is $\mathbb{C}[z_1,\ldots,z_n]^G$. From now on, we shall view $G\backslash\mathbb{C}^n$ as such. The germ of $G\backslash\mathbb{C}^n$ at $G.0$ (and any analytic germ isomorphic to such a germ) is called a *quotient singularity*. It is clear that $G\backslash\mathbb{C}^n$ is isomorphic to $\mathbb{C}^n$ if and only if $\mathbb{C}[z_1,\ldots,z_n]^G$ is a polynomial algebra. There is a beautiful characterization of such G due to Chevalley (1955): $\mathbb{C}[z_1,\ldots,z_n]^G$ is a polynomial algebra if and only if G is generated by complex reflections ($g \in GL_n(\mathbb{C})$ is called a *complex reflection* if it is of finite order $\neq 1$ and leaves a hyperplane pointwise fixed). (This reduces the study of quotient singularities $G\backslash\mathbb{C}^n$ to the case where G contains no complex reflections. For if $H \subset G$ denotes the subgroup generated by all complex reflections in G, then H is normal in G and the factor group G/H will act on $H\backslash\mathbb{C}^n$ - this corresponds to the natural action of G/H on $\mathbb{C}[z_1,\ldots,z_n]^H$ - and $G\backslash\mathbb{C}^n$ is identified with the orbit space $(G/H)\backslash\mathbb{C}^n$. It can be shown that G/H contains no complex reflection.) We consider the case when $G \subset SL_2(\mathbb{C})$ in more detail. Notice that then G doesn't contain complex reflections: if $g \in SL_2(\mathbb{C})$ leaves a line pointwise fixed, then it is unipotent and hence of infinite order unless $g = \mathbb{1}$. Since $G\backslash\mathbb{C}^2$ is normal, it will have G.0 as an isolated singular point if $G \neq \{\mathbb{1}\}$.

Example 2. Let m be a positive integer and let $G = \{\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix} : \zeta^m = 1\}$. Then $\mathbb{C}[z_1,z_2]^G$ is generated by z_1^m, z_2^m, z_1z_2. We use these as the coordinates of $f : \mathbb{C}^2 \to \mathbb{C}^3$. Clearly, the image of f is contained in the hypersurface defined by $t_1t_2-t_3^m = 0$ As this hypersurface is irreducible and $\dim f(\mathbb{C}^2) = 2$, the two must coincide. So the germ $(G\backslash\mathbb{C}^2, G.0)$ is isomorphic to the hypersurface singularity defined by $t_1t_2-t_3^m$. It is customary to denote the