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1 INTRODUCTION AND PRELIMINARIES

1.1 INTRODUCTION

The theory of analytic (one parameter) semigroups $t \mapsto a^t$ from the open right half plane H into a Banach algebra is the main topic discussed in these notes. Several concrete elementary classical examples of such semigroups are defined, a general method of constructing such semigroups in a Banach algebra with a bounded approximate identity is given, and then relationships between the semigroup and the algebra are investigated. These notes form small sections in the theory of (one parameter) continuous semigroups and in the general theory of Banach algebras. They emphasize an approach that is standard to neither of these subjects. A study of Hille and Phillips [1974] reveals that the theory of Banach algebras has been used as a tool in the study of certain problems in continuous semigroups, but that semigroup theory has until recently (1979) not impinged on the theory of Banach algebras. These lecture notes are about this recent progress.

Throughout these notes we use 'semigroup' for 'one parameter semigroup' when discussing a homomorphism from an additive subsemigroup of $\mathbb{C}$ into a Banach algebra, and we write our semigroups $t \mapsto a^t$ to emphasize the power law $a^{t+s} = a^t.a^s$ and function property of the semigroup. In the standard works on semigroups much attention is given to strongly continuous semigroups and their generators (see Hille and Phillips [1974], Dunford and Schwartz [1958], and Reed and Simon [1972]). In these works the generator itself is important, plays a fundamental role, and is often an object of considerable mathematical interest (for example, it may be the Laplacian). As the theory is developed here the generator is useful only in Chapter 6, and even there it is the resolvent $(1 - R)^{-1}$, not the generator R, that occurs in our Banach algebra results. It is possible in the Banach algebra situation to develop lemmas corresponding to the Hille-Yoshida Theorem totally avoiding unbounded closed operators and working with what is essentially the inverse of the generator. This seemed artificial and we do not do it here. In the standard works on semigroups (ibid.) most of the emphasis is on semigroups that are not quasinilpotent, and there is little or no space devoted to quasinilpotent semigroups (see Hille and Phillips [1974], p.481). However Chapters 5 and 6 of these notes concern radical Banach algebras, perhaps indirectly. In these radical algebras we are studying quasinilpotent semigroups.

The general theory of Banach algebras has mostly been developed for (Jacobson) semisimple algebras, and the most studied families of Banach algebras are semisimple: C*-algebras, group algebras, and uniform algebras. A brief glance through the standard references (Rickart [1960] and Bonsall and Duncan [1973]) illustrates this. Radical algebras and quasinilpotent elements play a very important role in Chapters 5 and 6 of these notes. However we do not attempt a study of radical Banach algebras or even discuss the role of non-continuous semigroups in the classification of radical Banach algebras. Various weaker assumptions on the domain of a semigroup $t \mapsto a^t$, for example, to the rational numbers, are related to the structure of certain radical Banach algebras (see Esterle [1980b]). Strongly continuous (one parameter) groups of automorphisms on a C*-algebra are fundamental in C*-algebra theory (see Pedersen [1979]). Except for this there had been few applications of semigroup theory to Banach algebras until 1979.

The standard references on the theory of semigroups (Hille and Phillips [1974], Dunford and Schwartz [1958], and Reed and Simon [1972]) contain much of Chapter 2 and the Hille-Yoshida Theorem of Chapter 6. The approach here is also basically different from that in Butzer and Berhens [1967], and Berge and Forst [1975]. The modification of the Cohen factorization theorem discussed in Chapters 3 and 4 is covered in considerable detail in Doran's and Wichman's lecture notes [1979] on bounded approximate identities and Cohen factorization. Even here our account differs from the original version, which is what they give.

These notes are elementary and the results are proved in detail. As background for the main results we assume standard elementary functional analysis, the complex analysis in Real and Complex Analysis by Rudin [1966], and the Banach algebra theory in Complete Normed Algebras by Bonsall and Duncan [1973]. We shall use the Titchmarsh convolution theorem (see Mikusiński [1959], Chapter 2) a couple of times. In a few corollaries and applications considerably more is assumed (for example, there are results applying to $L^1(G)$). Calculations are given in detail even when standard functions in $L^1(\mathbb{R})$ are being considered. The main tools in our proofs are techniques from Banach algebra theory and semigroup theory, the Bochner integral, and some classical results of complex analysis. Although the Hille-Yoshida and Ahlfors-Heins Theorems are standard results readily available in books, they are not in the assumed background and so they are proved in suitable forms in these notes (Theorem 6.7 and Appendix A1.1). In the introduction the Bochner integral is briefly discussed.

The notes are not polished. Each chapter beyond the first ends with notes and remarks where brief reference will be made to the literature, related results, and open problems. The bibliography is not comprehensive.

These notes are an expanded and revised version of lectures that I gave at the University of Edinburgh in January, February, and March 1980. The lectures and notes were both influenced by a course that J. Esterle gave in the University of California, Los Angeles, in April, May, and June 1979. Some parts of my lectures appear as they were given, others have been extensively revised, and occasionally a single verbal remark in a lecture has become a whole section here. The concrete semigroups in $L^1(\mathbb{R}^+)$ and $L^1(\mathbb{R}^n)$ were covered as here (Chapter 2) as was the Wiener Tauberian Theorem, (Theorem 5.6), Theorem 5.3, and the whole of Chapter 6. Chapters 3 and 4 were a single unproved result in lectures, but several of the audience had suffered talks from me on these subjects in a seminar.

I am grateful to many mathematicians for preprints and odd half forgotten conversations, which have influenced the development, and to the audience who survived my lectures. I am grateful to P.C. Curtis, Jr. and F.F. Bonsall for encouragement, to T.A. Gillespie for useful criticism of an early draft, to S. Grabiner for many discussions about Banach algebras, and to A.M. Davie for suggesting several improvements to results and proofs. H.G. Dales read the complete notes, and his detailed and careful criticism has enabled me to correct several errors and improve the notes. I am indebted to him for this and other suggestions. During 1978-9 J. Esterle and I had many discussions about radical Banach algebras and semigroups, and his U.C.L.A. lectures and seminars influenced my ideas. He has kindly given permission for me to include his results on nilpotent semigroups in Chapter 6 before he has published them. I am very grateful and deeply indebted to J. Esterle. Without his results in Chapters 5 and 6 these notes would not exist.

1.2 DEFINITIONS AND NOTATION

We shall now give some definitions, fix various notations, and prove a couple of useful little lemmas. Throughout these notes we shall consider complex Banach spaces and Banach algebras, and linear operators will be taken to be complex linear. The Banach algebras will not be assumed to have an identity, and these notes deal mainly with algebras without identity. If A is a Banach algebra, then $A \oplus \mathbb{C}1$ is the Banach algebra, obtained from A by formally adjoining an identity; note that the norm is $\|a + \lambda\| = \|a\| + |\lambda|$ for all $a \in A$ and $\lambda \in \mathbb{C}$. If A is a Banach algebra with identity $A^\# = A$, and if A is an algebra without identity $A^\# = A \oplus \mathbb{C}1$. The algebra $A^\#$ is the algebra in which the spectra of elements of A are calculated. The spectrum of $x \in A$ is denoted by $\sigma(x)$ and the spectral radius by $\nu(x)$.

If f is a function from a set X into a set Y, we shall often write $x \mapsto f(x) : X \to Y$. If X is a Banach space, $BL(X)$ denotes the Banach algebra of bounded linear operators on X. For a commutative Banach algebra A the <u>multiplier algebra</u> $Mul(A)$ is defined to be the set of $T \in BL(A)$ such that $T(ax) = aT(x)$ for all $x, a \in A$. Clearly $Mul(A)$ is a unital Banach algebra, and there is a natural norm reducing homomorphism $a \mapsto L_a : A \to Mul(A)$, where $L_a x = ax$ for all $x \in A$. Let Ω be a locally compact Hausdorff space and let $C_o(\Omega)$ be the Banach algebra of continuous complex valued functions on Ω vanishing at infinity. Then $C_o(\Omega)^\#$ is isomorphic to $C(\Omega \cup \{\infty\})$, where $\Omega \cup \{\infty\}$ is the one point compactification of Ω, and $Mul(C_o(\Omega))$ is isomorphic to $C(\beta\Omega)$, where $\beta\Omega$ is the Stone-Čech compactification of Ω.

Most of the Banach algebras we study have bounded approximate identities. A Banach algebra A has a <u>bounded approximate identity</u> Λ bounded by d if $\|f\| \le d$ for all $f \in \Lambda$, and if, for each finite subset $F \subseteq A$ and each $\varepsilon > 0$, there is an $e \in \Lambda$ such that $\|ea - a\| + \|ae - a\| < \varepsilon$ for all $a \in F$. If the set Λ can be chosen to be countable (commutative,...), we say that A has a countable (commutative,...) bounded approximate identity. If $\Lambda = \{a \in A : \|a\| \le d\}$ we shall suppress Λ. In Chapter 3 the countability of the bounded approximate identity is important. Here we note a couple of folklore facts which indicate that this hypothesis is not too restrictive for our purposes.

If a Banach algebra A is separable and has a bounded approximate identity, then A has a countable bounded approximate identity. This can be seen by choosing a countable dense subset $\{y_n\}$ of A, and then choosing a sequence (e_n) from the bounded approximate identity such that $\| e_n y_j - y_j \| + \|y_j e_n - y_j\| < n^{-1}$ for $1 \leq j \leq n$ and all n. The set $\{e_n : n \in \mathbb{N}\}$ is a countable bounded approximate identity in A.

If A is a Banach algebra with a bounded approximate identity and if Y is a separable subspace of A, then there is a separable Banach subalgebra B of A that contains Y and has a bounded approximate identity. Let $\{y_n\}$ be a countable dense subset of Y and choose a sequence (e_n) from the bounded approximate identity of A such that
$\| e_n y_j - y_j\| + \|y_j e_n - y_j\| < n^{-1}$ and $\|e_n e_j - e_j\| + \|e_j e_n - e_j\| < n^{-1}$
for $1 \leq j \leq n-1$ and all n. The Banach subalgebra B of A generated by $Y \cup \{e_n : n \in \mathbb{N}\}$ has the required properties.

If a commutative Banach algebra A has a bounded approximate identity bounded by 1, then the natural homomorphism $a \mapsto L_a : A \to \mathrm{Mul}(A)$ from A into the multiplier algebra is an isometric embedding from A onto a closed ideal in $\mathrm{Mul}(A)$.

The complex numbers, real numbers, integers, and positive integers are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, respectively. The open right half plane $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ is denoted by H, and the closed right half plane by $\bar{H}$. The reader will be reminded of this notation periodically.

A function f from an open subset U of $\mathbb{C}$ into a Banach space X is said to be <u>analytic</u> if for each $z \in U$ the limit $\lim_{h \to 0} h^{-1}(f(z+h) - f(z))$ exists in X. This limit is denoted by $(Df)(z)$. The Hahn-Banach Theorem may be combined with results of complex analysis to yield results about analytic functions into a Banach space. The following result illustrates this technique and shows why it is not necessary to consider separately semigroups which are weakly or strongly analytic.

1.3 <u>LEMMA</u>

Let f be a function from an open subset U of the complex plane into a Banach space X. Then conditions (a), (b), and (c) on f are equivalent.

(a) f is analytic.

(b) Ff is analytic for all $F \in X^*$

(c) f is continuous, and

$$f(z) = \frac{1}{2\pi i} \int_\gamma f(\xi)\,(\xi - z)^{-1}\, d\xi,$$

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for each $z \in U$ and each closed path γ in U such that the winding number of z with respect to γ is 1 and of each point in $\mathbb{C} \setminus U$ is 0.

(d) If $X = BL(Y)$ for a Banach space Y, then the above conditions are equivalent to $z \mapsto F(f(z)y) : U \to \mathbb{C}$ being analytic for all $y \in Y$ and $F \in Y^*$.

<u>Proof</u> Clearly (a) implies (b) and (d).

If (b) holds and if $z, z_n \in U$ with $z_n \to z$, then $F((z_n - z)^{-1}(f(z_n) - f(z)))$ converges in $\mathbb{C}$ for each $F \in X^*$. The uniform boundedness theorem implies that the sequence $(\|(z_n - z)^{-1}(f(z_n) - f(z))\|)$ is bounded. Thus f is continuous on U. The integral in (c) now converges in X (see 1.6), and using the classical Cauchy integral formula for a complex valued analytic function we obtain

$$F(f(z)) = \frac{1}{2\pi i} \int_\gamma (Ff)(\xi)(\xi - z)^{-1}\, d\xi$$

$$= F\left(\frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - z)^{-1}\, d\xi\right)$$

for all $F \in X^*$. An application of the Hahn-Banach Theorem gives the equality of (c).

Now suppose that (c) holds. Let $z \in U$ and let γ be a small circle with centre z and radius r so that γ and its interior are contained in U. Let $M = \sup\{\|f(\xi)\| : \xi \in \gamma\}$. If $h \in \mathbb{C}$ with $|h| < r/2$, then

$$f(z + h) - f(z) - \frac{h}{2\pi i} \int_\gamma f(\xi)(\xi - z)^{-2}\, d\xi$$

$$= \frac{h^2}{2\pi i} \int_\gamma f(\xi)\ (\xi - z)^{-2}\ (\xi - z - h)^{-1}\, d\xi$$

so that

$$\left\| f(z+h) - f(z) - \frac{h}{2\pi i}\int_\gamma f(\xi)(\xi - z)^{-2}\, d\xi \right\|$$

$$\leq \frac{|h|^2}{2\pi} \cdot 2\pi r \cdot \frac{1}{r^2} \cdot \frac{1}{r/2} \cdot M.$$

This shows that f has derivative $\frac{1}{2\pi i}\int_\gamma f(\xi)\ (\xi - z)^{-2}\, d\xi$ at z, as we would expect from classical complex analysis. Hence (a) holds.

Suppose that (d) holds. Using the equivalence of (a) and (b) we find that $z \mapsto f(z)y : U \to Y$ is analytic for each $y \in Y$. An application of the uniform boundedness theorem as above shows that f is continuous on U. The integral in (c) now exists, and the equality in (c) is obtained by an application of the Hahn-Banach Theorem. The proof is complete.

A (one-parameter) <u>semigroup</u> in a Banach algebra A is a function $t \mapsto a^t$ from an additive subsemigroup of $\mathbb{C}$ containing the open right half line $(0,\infty)$ into A satisfying $a^{t+r} = a^t.a^r$ for all t,r in the domain of definition. We shall be concerned mainly with semigroups defined on the open right half plane H and on the open right half line $(0,\infty)$. The semigroup is said to be analytic or continuous (in some topology on A) if the function is analytic or continuous. Lemma 1.3 shows why we restrict attention to the norm topology when considering analytic semigroups defined on H. A semigroup $t \mapsto a^t$ is said to be a <u>contraction</u> semigroup if $\|a^t\| \leq 1$ for all t.

1.4 <u>LEMMA</u>

Let $t \mapsto a^t : H \to A$ be an analytic semigroup from the open right half plane into a Banach algebra A. Then $(a^t A)^- = (a^1 A)^-$ and $(A a^t)^- = (A a^1)^-$ for all $t \in H$.

<u>Proof</u>. Let $F \in A^*$ with $F(a^1 A) = \{0\}$. Then $t \mapsto F(a^t x) : H \to \mathbb{C}$ is an analytic function for each $x \in A$. This function is zero for all $t \in \mathbb{C}$ with $\operatorname{Re} t > 1$ because $F(a^t x) = F(a^1 a^{t-1} x) = 0$. Hence $F(a^t x) = 0$ for all $t \in H$ and $x \in A$. By the Hahn-Banach Theorem it follows that $(a^t A)^- \subseteq (a^1 A)^-$, and a similar argument yields the reverse inclusion.

Semigroups $t \mapsto a^t : (0,\infty) \to A$ that are not analytic may have

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$(a^t A)^-$ strictly decreasing, and Chapter 6 is devoted to the study of such semigroups. However even there our aim is to study continuous rather than strongly continuous semigroups.

1.5 AN OUTLINE OF THE NOTES

This section is a synopsis of the notes. In Chapter 2 examples of analytic semigroups from the open right half plane into several concrete Banach algebras are discussed. These semigroups will suggest abstract properties to be investigated in Chapter 3 and will provide examples for the results and proofs of Chapter 5. The structure of C^*-algebras is very rich, and this makes it easy to construct analytic semigroups in C^*-algebras by using the commutative Gelfand-Naimark Theorem to define a^t for a positive element in the algebra and t in H. However these semigroups in C^*-algebras are not useful tools to study the algebra. Semigroups in various convolution algebras are more interesting than in C^*-algebras and throw more light on the structure of the algebra. The fractional integral semigroup $t \mapsto I^t$, where $I^t(w) = \Gamma(t)^{-1} w^{t-1} e^{-w}$, and the backward heat semigroup $t \mapsto C^t$, where $C^t(w) = t(2\pi^{\frac{1}{2}})^{-1} w^{-3/2} \exp(-t^2/4w)$, are given in detail as examples of semigroups into $L^1(\mathbb{R}^+)$ (see Theorem 2.6, and Lemma 2.9).

The Banach algebra $L^1(\mathbb{R}^+)$ is very important in the study of semigroups in Banach algebras for the following reason. If $t \mapsto a^t : (0,\infty) \to A$ is a continuous contraction semigroup into a Banach algebra A, then there is a natural norm reducing homomorphism Θ from $L^1(\mathbb{R}^+)$ into A defined by $\Theta(f) = \int_0^\infty f(t)a^t\, dt$. The integral is a Bochner integral (1.6), the norm reducing property $(\|\Theta\| \leq 1)$ follows from $\|a^t\| \leq 1$ for all $t > 0$, and the homomorphism property follows by a change in the order of integration. The homomorphism Θ may be used to map analytic semigroups in $L^1(\mathbb{R}^+)$ into analytic semigroups in A. The image semigroups are called <u>subordinate</u> to the semigroup $t \mapsto a^t$. The <u>Gaussian</u> semigroup

$$G^t(w) = (4\pi t)^{-n/2} \exp(-|w|^2/4t)$$

and <u>Poisson</u> semigroup

$$P^t(w) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \cdot \frac{t}{(t^2 + |w|^2)^{(n+1)/2}}$$

in $L^1(\mathbb{R}^n)$ are discussed, and the Poisson semigroup is studied via its subordination to the Gaussian semigroup. Certain growth properties of $\|G^t\|_1$ are substantially better than those of $\|I^t\|_1$. These strong growth properties of $\|G^t\|_1$ are crucial in the proof of the Wiener Tauberian Theorem (Theorem 5.6).

Chapter 3 contains a theorem that gives the existence of analytic semigroups in a Banach algebra with a countable bounded approximate identity. The result is proved by modifying the proof of Cohen's factorization theorem so that the proof resembles the way in which a strongly continuous semigroup is generated in the Hille-Yoshida Theorem (see 6.7). The semigroup constructed in Chapter 3 has growth and structure more like the fractional integral semigroup than the Gaussian semigroup. The general semigroup result is applied to the group algebra $L^1(G)$ of a metrizable locally compact group, and to obtain commutative bounded approximate identities in Banach algebras with countable bounded approximate identities.

The proof of the main result in Chapter 3 and the lemmas required in the proof fill Chapter 4. Two of the properties (6 and 15) of Theorem 3.1 I have not been able to prove by the exponential methods of Chapter 4. These two properties require the factorization results developed in Sinclair [1979a], although I believe they may be obtained from exponential calculations. I have attempted to prove the most general factorization result for semigroups that I know. In other chapters generality has often been sacrificed to obtain an elementary account.

The properties of the semigroups near the boundaries of their domains of definition are interesting, and are closely related to the fine structure of the semigroup and the algebra. In Chapter 5 we investigate the restrictions imposed on a commutative Banach algebra A by the assumption that it contains an analytic semigroup $t \mapsto a^t : H \to A$ with $(a^1 A)^- = A$ such that the growth of $\|a^t\|$ is suitably restricted. The restrictions we consider are:

(i) to the growth of $\|a^t\|$ along rays in H emanating from 0 (5.2);

(ii) to the growth of $\|a^t\|$ along a vertical line (5.5);

(iii) the boundedness in the semidisc $\{z \in H: |z| \le 1\}$ (5.12).

In each of these cases we prove a result due to Esterle : these are, respectively, a result about radical Banach algebras, a Tauberian theorem, and a result on the non-separability of the multiplier algebra. The philosophy underlying the proofs in the first two cases is to define a suitable classical analytic function F (by using the analyticity of the semigroup and a continuous linear functional on the algebra), and to apply the Ahlfors-Heins Theorem to F.

Chapter 6 begins with a standard account of the Hille-Yoshida Theorem relating a strongly continuous contraction semigroup a^t on a Banach space with the closed operator R that is its infinitesimal generator. The relationships between the nilpotency of the semigroup and the growth of $\|(1 - R)^{-n}\|$ as n tends to infinity is investigated. This result is applied to a hyperinvariant subspace theorem for suitable operators on a Banach space, and to prove the existence of a proper closed ideal in a commutative radical Banach algebra containing a non-zero element u such that $\|u(\lambda - u)^{-1}\| \leq 1$ for all $\lambda > 0$ and $\{n\|u^n\|^{1/n} : n \in \mathbb{N}\}$ is bounded. This process is seen to give abstractly the obvious ideals in the Volterra convolution algebra $L^1_*[0,1]$.

In the appendix we give a proof of a special case of the Ahlfors-Heins Theorem, and prove a theorem of G.R. Allan [1979] on certain closed ideals of $L^1(\mathbb{R}^+,\omega)$ for ω a radical weight. It is also shown that an Arens regular Banach algebra with a bounded approximate identity has a bounded approximate identity that is well behaved with respect to derivations (and is quasicentral).

1.6 INTEGRALS

We shall frequently integrate continuous functions on $(0,\infty)$ with values in a Banach space, and in this section we briefly define the elementary integrals used. There are extensive discussions of the integration of Banach space valued functions in Dunford and Schwartz [1958] and in Hille and Phillips [1974].

Let f be a continuous function from $(0,\infty)$ into a Banach space X with $\int_0^\infty \|f(w)\|\,dw$ finite. For $0 < \alpha < \beta < \infty$ we define $\int_\alpha^\beta f(w)\,dw$ as the limit of Riemann sums using partitions $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta$ with $\max\{\alpha_j - \alpha_{j-1} : 1 \leq j \leq n\}$ tending to zero. The limit may be shown to exist in X in the same way that the classical Riemann integral is shown to exist by using the uniform continuity of the integrand. Further the integral $\int_\alpha^\beta f(w)\,dw$ is seen to be linear in f, and to satisfy $\left\|\int_\alpha^\beta f(w)\,dw\right\| \leq \int_\alpha^\beta \|f(w)\|\,dw$ and $F\left(\int_\beta^\alpha f(w)\,dw\right) = \int_\alpha^\beta (Ff)(w)\,dw$ for all $F \in X^*$. Using the observation that $\int_0^\alpha + \int_\beta^\infty \|f(w)\|\,dw$ tends to zero as α tends to