THE ASYMPTOTIC SPEED AND SHAPE OF A PARTICLE SYSTEM*

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1 Introduction

We study in this paper the asymptotics as $k \to \infty$ of the motion of a system of k particles located at sites labelled by the integers. This section gives an informal description of the particle system and our results, and the original motivation for the study.

The particles will be referred to as *balls*, and the sites as *boxes*. The motion may be described as follows. Initially the k balls are distributed amongst boxes in such a way that the set of occupied boxes is *connected*. (A box may contain many balls, but there is no empty box between two occupied boxes.) At each move, a ball is taken from the left-most occupied box and placed one box to the right of a ball chosen uniformly at random from among the k balls, the successive choices being mutually independent. It is clear that the set of occupied boxes remains connected, and that the collection of balls drifts off to infinity. It is easy to see that for each k the k-ball motion drifts off to infinity at an almost certain average speed s_k, defined formally by (2.3) below. Our main result is that $s_k \sim e/k$ as $k \to \infty$. To be more precise:

THEOREM 1.1 *As* k *increases to infinity,* ks_k *increases to* e.

This result was conjectured by Tovey (private communication), and informal arguments supporting the conjecture have been given by Keller (1980) and Weiner (1980). Our method of proof (Sections 2-4) is to use coupling to compare the k-ball process with a certain, more easily analysed, pure growth process (defined at (3.3)).

Secondly, for fixed k we can define (Section 5) a random vector $(\pi_0, \pi_1, \pi_2, \ldots)$ describing the equilibrium proportions of balls in the (0th,

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1st, 2nd,...) box from the leftmost occupied box at times when the leftmost box has just been cleared. We conjecture (5.9) that as $k \to \infty$, $(\pi_0, \pi_1, \pi_2, \ldots)$ converges in distribution to a certain sequence $(\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots)$ of constants with $\hat{p}_0 = e^{-1}$. This would imply that for large k the process of proportions in the k-ball process evolves almost deterministically. In Section 6 we show how this conjecture is related to problems concerning a certain transformation of probability measures on the positive integers.

The origin of the k-ball process came in work of Tovey (1980) on abstractions of local improvement algorithms. Consider functions f defined on the vertices of the d-dimensional cube, with distinct real values, and with the *local-global* property:

f has no local maximum except the global maximum.

(Here a *local* maximum is a vertex i such that $f(i) > f(j)$ for each *neighbor* j of i, and vertices are *neighbors* if they are connected by an edge.) There is an obvious algorithm to locate the maximum of a local-global function: move from a vertex v to the neighbor v' for which $f(v')$ is largest; unless v is a local maximum, in which case it must be the global maximum. How good is this algorithm "on average"? In other words, what is the expected number of steps required to locate the maximum of a function f picked at random according to some distribution μ on local-global functions? Now any function f induces an ordering $v_1, v_2, v_3, \ldots$ of vertices such that $f(v_1) > f(v_2) > f(v_3) > \ldots$, and f is local-global iff

$$v_n \text{ is a neighbor of at least one of } \{v_1, \ldots, v_{n-1}\},\ n \ge 2. \quad (1.2)$$

Thus a distribution μ on local-global functions induces a random ordering $V_1, V_2, V_3, \ldots$ of vertices satisfying (1.2). Let N_i denote the number of steps required by the algorithm started at vertex V_i, so that $\max_i N_i$ is the number of steps required from the "worst" initial vertex. Plainly (N_i) satisfies the recurrence

$$\begin{aligned} N_1 &= 1 \\ N_{i+1} &= 1 + N_j, \text{ where } j \le i \text{ is the least integer for which} \\ &\qquad V_j \text{ is a neighbor of } V_i \end{aligned} \quad (1.3)$$

But we can think of (N_i) as a process of balls in boxes, where ball i corresponds to vertex V_i and the box containing ball i corresponds to the number N_i of steps in the algorithm. Then (1.3) says the process evolves by the $(i+1)^{st}$ ball being placed in the box to the right of the box containing ball j, where j is chosen in some random way from the existing balls. And $\max_i N_i$, the number of steps required from the worst initial vertex, is the position of the rightmost occupied box after 2^d balls have been used. To estimate this, we need an upper bound on the speed of the rightmost occupied box of this new "balls in boxes" process. This process differs from the k-ball process we study-- because j is chosen in some complicated random way involving the distribution μ on local-global functions, and the number of balls increases. But coupling arguments in the spirit of those of Section 3 can be used to show that for certain distributions μ the speed of the rightmost occupied box of the new process is less than the speed of the k-ball process: see Tovey (1980).

2. Preliminaries

The configuration of a finite number of balls in boxes numbered 0,1,2,... will be described by a k-tuple of nonnegative integers

$$\underset{\sim}{x} = (x_1, x_2, \ldots, x_k)$$

where $k = \#\underset{\sim}{x}$ is the total number of balls; the balls are assumed to be labelled $1,\ldots,k$, and x_j is the box number of ball number j. The distribution of balls among boxes without regard to labelling is recorded by the *counting measure* $\underset{\sim}{N}\underset{\sim}{x} = (N_i\underset{\sim}{x},\ i=0,1,\ldots)$ defined by

$$N_i\underset{\sim}{x} = \#\{j:\ x_j = i\}, \quad i=0,1,\ldots .$$

So $N_i\underset{\sim}{x}$ is the number of balls in box i for configuration $\underset{\sim}{x}$. The *left end* $L\underset{\sim}{x}$ and *right end* $R\underset{\sim}{x}$ of configuration $\underset{\sim}{x}$ are defined by

$$L\underset{\sim}{x} = \min_j x_j = \min\{i:\ N_i\underset{\sim}{x} > 0\},$$

$$R\underset{\sim}{x} = \max_j x_j = \max\{i:\ N_i\underset{\sim}{x} > 0\},$$

and $\underset{\sim}{x}$ is *connected* if

$$N_i \underset{\sim}{x} > 0 \text{ for } L\underset{\sim}{x} < i < R\underset{\sim}{x} .$$

The set of connected configurations of k balls will be denoted C_k, and we put $C_* = \cup_k C_k$. For $x \in C_k$, $i = 1,\ldots,k$, define a new configuration $\underset{\sim}{x}^i$ by

$$\begin{aligned} x_j^i &= x_j \qquad \text{except if } j = \hat{j} \\ &= x_i + 1 \qquad \text{if } j = \hat{j} \end{aligned} \tag{2.1}$$

where $\hat{j} = \hat{j}(\underset{\sim}{x})$ is the number of the lowest numbered ball in the left end box $L\underset{\sim}{x}$. That is to say, $\underset{\sim}{x}^i$ is obtained from $\underset{\sim}{x}$ by removing ball $\hat{j}$ from box $L\underset{\sim}{x}$ and replacing it in the box to the right of ball i. Clearly $\underset{\sim}{x}^i \in C_k$ for all $i = 1,\ldots,k$. The *discrete* k-*ball process* is the discrete time Markov chain with countable state space C_k and one step transition matrix $p_k(\underset{\sim}{x},\underset{\sim}{y})$ defined by

$$p_k(\underset{\sim}{x},\underset{\sim}{x}^i) = 1/k , \quad i = 1,\ldots,k . \tag{2.2}$$

The *speed* s_k of the discrete k-ball process is the constant

$$s_k = \lim_{m\to\infty} m^{-1} L\underset{\sim}{X}(m) = \lim_{m\to\infty} m^{-1} R\underset{\sim}{X}(m) , \tag{2.3}$$

where $(\underset{\sim}{X}(m), m = 0,1,\ldots)$ is a discrete k-ball process. We assert that the limits exist almost surely and do not depend on the initial configuration $\underset{\sim}{X}(0)$. To see why, consider the *left counting process*

$$(\underset{\sim}{N}^L \underset{\sim}{X}(m), m = 0,1,\ldots)$$

where for a configuration $\underset{\sim}{x}$ the *left count of* $\underset{\sim}{x}$ is the vector $\underset{\sim}{N}^L \underset{\sim}{x} = (N_i^L \underset{\sim}{x}, i = 0,1,\ldots)$ defined by

$$N_i^L \underset{\sim}{x} = N_{L\underset{\sim}{x}+i} \underset{\sim}{x} , \quad i \geq 0 . \tag{2.4}$$

Indeed, for given k the left counting process is a Markov chain whose *finite* state space is the set of counting vectors $\underset{\sim}{n} = (n_i, i \geq 0)$ such $\sum_i n_i = k$ and there exists $d \geq 1$ with

$$n_i > 0 \text{ for } i \leq d, \; n_i = 0 \text{ for } i > d .$$

This motion is easily seen to be irreducible and aperiodic, so there is a unique equilibrium distribution, λ_k say. Now from the definition of the k-ball process, for $m \geq 1$ the random variable $L\underset{\sim}{X}(m) - L\underset{\sim}{X}(m-1)$ is identical to the indicator of the event $(N_0^L\underset{\sim}{X}(m-1)=1)$, which we shall refer to as a *clearance at move* m. Thus

$$\lim_{m\to\infty} m^{-1} L\underset{\sim}{X}(m) = \lim_{m\to\infty} m^{-1} \#\{j \leq m : N_0^L \underset{\sim}{X}(j)=1\} \tag{2.5}$$

$$= \lambda_k(\underset{\sim}{n}: n_0=1) \text{ a.s.}$$

This justifies (2.3), since obviously

$$0 \leq R\underset{\sim}{X}(m) - L\underset{\sim}{X}(m) \leq k .$$

There are several other expressions for the speed s_k: Section 7 describes some we do not use, but let us here give only one, which will be the basis of developments in Section 5. Recall that for an irreducible Markov chain $Y(0), Y(1), \ldots$ with equilibrium distribution λ, if $Y(0)$ is given the distribution $\lambda|A$ obtained by conditioning λ on a set of states A, then

(i) the return time

$$T_A = \inf\{m: Y_m \in A\}$$

has expectation $1/\lambda(A)$, and

(ii) the distribution of $Y(T_A)$ is $\lambda|A$.

See, for example, Freedman (1971), Section 2.5. Applying this fact to $Y(m) = \underset{\sim}{N}(m)$ and $A = (\underset{\sim}{n}: n_0=1)$, the definition of the k-ball process implies that T_A is identical to $N_0(1)$. Let ν_k be the distribution of $\underset{\sim}{N}(1)$ when $\underset{\sim}{N}(0)$ has distribution $\lambda_k|(n: n_0=1)$, and call ν_k the *clearance equilibrium* of the left counting process. After a change of variables, (i) above in conjunction with (2.5) yields the formula

$$1/s_k = E_{\nu_k} N_0 , \tag{2.6}$$

where the right side denotes the expectation of N_0 when N has distribution ν_k. We record also for later use a consequence of (ii) above. For the left counting process $(\underset{\sim}{N}(0), \underset{\sim}{N}(1),\ldots)$,

$$\text{if } \underset{\sim}{N}(0) \text{ has distribution } \nu_k, \text{ then so does } \underset{\sim}{N}(N_0(0)) . \tag{2.7}$$

For an arbitrary initial distribution, the distribution ν_k can still be interpreted as the limiting distribution of $\underset{\sim}{N}(M_n)$ as $n \to \infty$, where M_n is the time of the n^{th} clearance:

$$M_{n+1} = M_n + N_0(M_n), \quad n \geq 0, \quad M_0 = 0.$$

Further, ν_k is the almost sure limit as $n \to \infty$ of the empirical distribution of the sequence $\underset{\sim}{N}(M_1), \underset{\sim}{N}(M_2),\ldots,\underset{\sim}{N}(M_n)$.

3. Speed comparisons

To facilitate comparison of the speeds s_k for different values of k, we introduce now the *continuous k-ball process*. This is the Markov process with countable state space C_k and continuous time parameter $t \geq 0$ which is specified by the transition rates.

$$\underset{\sim}{x} \to \underset{\sim}{x}^i, \text{ rate } 1, \; i=1,\ldots,k . \tag{3.1}$$

Here, and later in similar descriptions of transition rate matrices, off-diagonal rates not explicitly mentioned are assumed to be zero, and the diagonal entries are taken to make the row sums zero. Put another way, $(\underset{\sim}{X}_t, t\geq 0)$ is a continuous k-ball process iff

$$\underset{\sim}{X}_t = \underset{\sim}{Y}_{M(t)} , \quad t \geq 0$$

where $\underset{\sim}{Y}_0, \underset{\sim}{Y}_1,\ldots$ is a discrete k-ball process and $(M(t), t\geq 0)$ is an independent Poisson process with rate k.

Since $t^{-1}M(t) \to k$ a.s., a comparison with (2.3) above shows that the continuous k-ball process has speed

$$\lim_{t\to\infty} L\underset{\sim}{X}_t/t = \lim_{t\to\infty} R\underset{\sim}{X}_t/t = ks_k \ . \qquad (3.2)$$

Notice that both $L\underset{\sim}{X}_t$ and $R\underset{\sim}{X}_t$ are determined by the counting measure $N\underset{\sim}{X}_t$, so the speed is determined by the counting measure process $(N\underset{\sim}{X}_t,\ t\geq 0)$. Now so far as these counts are concerned, one can view the transition $\underset{\sim}{x} \to \underset{\sim}{x}^j$ of a k-ball process as the creation of a new ball in box x_j+1, together with simultaneous annihilation of a ball in the left-most occupied box $L\underset{\sim}{x}$. From this point of view the motion proceeds as if each ball were splitting at rate 1 into two balls, independently of other balls. These two balls are a "mother" ball remaining in the original box and a "daughter" ball appearing one box to the right, with annihilation of one ball in the leftmost occupied box simultaneous with each split. This suggests comparing the continuous k-ball process with the Markov process whose state space is $C_* = \cup_k C_k$ which evolves according to the splitting rules described above but with no annihilations. We call this the *lateral birth process*. Its transition rates are

$$\underset{\sim}{x} \to \underset{\sim}{x}^{i+} \text{ at rate } 1, \quad i=1,\ldots,\#\underset{\sim}{x} \ , \qquad (3.3)$$

where $x^{i+} \in C_{\#x+1}$ is defined by

$$x_j^{i+} = x_j \quad , \quad j=1,\ldots,\#\underset{\sim}{x}$$
$$= x_i + 1, \quad j = \#\underset{\sim}{x}+1 \ .$$

In the terminology of Mollison (1978), the lateral birth process is a particularly simple contact birth process. The next result is a special case of more general results for branching processes and Markovian contact processes due to Kingman (1975) and Mollison (1978), but for the sake of completeness we shall provide a proof in Section 4.

PROPOSITION 3.4. *For a lateral birth process* $(\underset{\sim}{B}_t,\ t\geq 0)$,

$$\lim_{t\to\infty} R\underset{\sim}{B}_t/t = e \text{ a.s.}$$

To compare the progress of different processes of balls in boxes

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we introduce a partial ordering of configurations. Say $\underset{\sim}{x}$ is behind $\underset{\sim}{y}$, or $\underset{\sim}{y}$ is ahead of $\underset{\sim}{x}$, and write $\underset{\sim}{x} \le \underset{\sim}{y}$ iff

$$\sum_{j\ge h} x_j \le \sum_{j\ge h} y_j \ , \quad h = 0,1,\ldots .$$

So $\underset{\sim}{x} \le \underset{\sim}{y}$ iff $\underset{\sim}{y}$ has more balls than $\underset{\sim}{x}$ to the right of h for each box h. (To avoid ambiguity we say box i is to the right of box h iff $i \ge h$, strictly to the right of box h iff $i > h$, and just to the right of h iff $i = h+1$, with a similar convention on the left.) Given a configuration $\underset{\sim}{x}$, let the balls of $\underset{\sim}{x}$ be ranked primarily according to their position, and secondarily according to their label. That is to say, the rank of ball j in configuration $\underset{\sim}{x}$ is one plus the number of balls in boxes strictly to the right of ball j plus the number of balls in the same box as ball j whose labels exceed j. For each configuration $\underset{\sim}{x}$ this gives a total ordering of the balls comprising $\underset{\sim}{x}$. It is easy to see that $\underset{\sim}{x} \le \underset{\sim}{y}$ iff $\#\underset{\sim}{x} \le \#\underset{\sim}{y}$ and for each $r = 1,2,\ldots,\#\underset{\sim}{x}$, the ball of rank r in $\underset{\sim}{x}$ is to the left of the ball with rank r in $\underset{\sim}{y}$. In particular, taking $r = 1$ shows that $\underset{\sim}{x} \le \underset{\sim}{y}$ implies $R\underset{\sim}{x} \le R\underset{\sim}{y}$. But, $\underset{\sim}{x} \le \underset{\sim}{y}$ does not imply $L\underset{\sim}{x} \le L\underset{\sim}{y}$, except when $\#\underset{\sim}{x} = \#\underset{\sim}{y}$.

Consider now two continuous time Markov chains M and $\hat{M}$ with state spaces C and $\hat{C}$ which are subsets of C_*, and bounded transition rate matrices Q and $\hat{Q}$ indexed by C and $\hat{C}$ respectively. Say M stays behind $\hat{M}$ (or $\hat{M}$ stays ahead of M) if for every pair of initial configurations $\underset{\sim}{x}$ and $\hat{\underset{\sim}{x}}$ with $\underset{\sim}{x} \le \hat{\underset{\sim}{x}}$ there exists an M-chain $(\underset{\sim}{X}_t,\ t\ge 0)$ and an $\hat{M}$-chain $(\hat{\underset{\sim}{X}}_t,\ t\ge 0)$ defined on the same probability space such that

$$\underset{\sim}{X}_0 = \underset{\sim}{x},\ \hat{\underset{\sim}{X}}_0 = \hat{\underset{\sim}{x}}, \quad \text{and}$$

$$\underset{\sim}{X}_t \le \hat{\underset{\sim}{X}}_t \quad \text{for all} \quad t \ge 0.$$

We shall make use of the following Lemma, which is valid for Markov chains on an arbitrary partially ordered countable set C_*. (The idea here is folklore amongst coupling theorists; Liggett (1977) applies the same idea with a different partial ordering in investigating infinite particle systems.)

LEMMA 3.6. *Suppose that* $\hat{x} \le \hat{y}$ *whenever* $\hat{Q}(\hat{x},\hat{y}) > 0$. *Suppose also that for every pair of states* $(x,\hat{x}) \in C\times\hat{C}$ *with* $x < \hat{x}$ *and every state* y *with* $Q(x,y) > 0$ *there is a state* $\hat{y} = f(x,\hat{x},y)$ *such that*

(i) $y \le \hat{y}$

(ii) $Q(x,y) \le Q(\hat{x},\hat{y})$

(iii) *For fixed* x *and* $\hat{x}$ *the map* $y \to \hat{y}$ *is one-to-one.*

Then $\hat{M}$ *stays ahead of* M.

PROOF. Construct $(X,\hat{X})$ as the Markov chain on $C\times\hat{C}$ with transition rates

$$\begin{aligned}(x,\hat{x}) &\to (y,\hat{y}),\ \text{rate } Q(x,y)\\ &\to (x,\hat{y}),\ \text{rate } \hat{Q}(\hat{x},\hat{y}) - Q(x,y)\\ &\to (x,y'),\text{rate } \hat{Q}(\hat{x},y')\ ,\end{aligned}$$

where $\hat{y} = f(x,\hat{x},y)$ and y' is an arbitrary state not in the range of $f(x,\hat{x},\cdot)$. These transitions stay within the set $\{x\le\hat{x}\}$.

REMARK 3.7. In the applications below, (ii) holds with equality. The process $(X,\hat{X})$ above can then be described more simply by saying that $\hat{X}$ is an $\hat{M}$-chain and that X is derived from $\hat{X}$ by letting X make a transition from x to y iff $\hat{X}$ makes a transition from $\hat{x}$ to $\hat{y} = f(x,\hat{x},y)$.

PROPOSITION 3.8. *For each* k *the continuous* k-*ball process stays*

(i) *behind the continuous* $\hat{k}$-*ball process if* $k < \hat{k}$,

(ii) *behind the lateral birth process,*

(iii) *ahead of the lateral birth process, stopped at the time* T_k *that it first attains size* k.

PROOF. These are simple applications of the Lemma. In each case the mapping $\hat{y} = f(x,\hat{x},y)$ is defined like this: if y is obtained from x by putting a ball just to the right of the ball ranked r in x, where $1 \le r \le \#x$, then $\hat{y}$ is obtained from $\hat{x}$ by putting a ball just to the right of the ball ranked r in $\hat{x}$.

Let σ_k now denote the speed of the continuous k-ball process, so from (3.2) we have $\sigma_k = ks_k$ where s_k is the speed of the discrete k-ball process.

COROLLARY 3.9.

(i) $\sigma_k \le \sigma_{\hat{k}}$ if $k \le \hat{k}$.

(ii) $\sigma_k \le e$

(iii) $\sigma_k \ge ERB(T_k)/ET_k$,

where $(B(t),\ t\ge 0)$ *is a lateral birth process starting with a single ball in box* 0, *and* $T_k = \inf\{t:\ \#B(t)=k\}$.

PROOF. In view of (3.4), (i) and (ii) follow at once from the corresponding parts of the proposition. For (iii), consider a continuous k-ball process $(X(t),\ t\ge 0)$. By (3.8)(ii) we can construct a process $(B_1(t):\ 0\le t\le T_{1k})$ such that

$B_1(t) \le X(t),\quad 0 \le t \le T_{1k}$,

$B_1(0)$ is the configuration with one ball in box $RX(0)$ and no other balls,

B_1 evolves as a lateral birth process run until the time T_{1k} it first reaches size k.

Now repeat the construction to obtain a second lateral birth process $(B_2(t):\ 0 \le t \le T_{2k})$ such that

$B_2(t) \le X(T_{1k}+t),\ 0 \le t \le T_{2k}$

$B_2(0)$ is the configuration with one ball in box $RX(T_{1k})$ and no other balls,

B_2 evolves as a lateral birth process run until the time T_{2k} it first reaches size k.

Repeating indefinitely, one obtains a sequence of stopped lateral birth processes $B_1, B_2, \ldots$ running behind portions of the k-ball process. To be precise

$$B_n(t) \le X(S_{n-1}+t),\quad 0 \le t \le T_{nk}\ , \tag{3.9}$$

where $S_n = T_{1k} + \ldots + T_{nk}$, and $B_n(0)$ is a single ball in box

$$RB_n(0) = RX(S_{n-1})\ . \tag{3.10}$$