INTRODUCTION

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The last 15 years have seen the creation of a new branch of mathematics: the theory of 'integrable' non-linear partial differential equations. The prototype example is the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx}, \quad u = u(x, t). \tag{1}$$

The theory was at first developed mainly by mathematical physicists, but recently it has been attracting the attention of an increasing number of mathematicians too. The involvement of mathematicians has been particularly strong in the Soviet Union; the articles reprinted in this volume contain fundamental contributions from some of the leading Soviet workers in the field. As befits pioneering works, the articles do not always make easy reading; in this introduction I shall try to smooth the reader's path by setting out the central facts about the KdV equation. For brevity I shall refer to the articles by the authors' initials [GD], [DMN], [K]. (I shall not discuss the other two articles, by Vinogradov–Krasilshchik and Vinogradov–Kupershmidt, which are of a different nature, nor the two short notes by Yusin and Krichever.) The notation for references is as follows: [GD2], for example, means reference 2 in the bibliography to the article of Gel'fand and Dikii; a reference without letters, such as [2], refers to the bibliography at the end of the introduction.

The main results in the theory of the KdV equation can reasonably be divided into three classes.

A. Formal (algebraic) properties of the equation (local conservation laws, Hamiltonian formalism).

B. Integration of the equation on certain infinite-dimensional function spaces (periodic, or rapidly decreasing at infinity).

C. Integration of the equation on certain finite-dimensional function spaces (the spaces of solutions of the 'higher stationary equations').

We begin with (A). Although the basic facts here are of an algebraic nature, their significance will be understood most easily if we first state their consequences in an analytic context. So let $\mathscr{F}$ denote one of the function spaces in

(B) above; more precisely, $\mathscr{F}$ is either (a) the space of C^∞ periodic functions (of one variable x) with some fixed period; or (b) some space of functions defined on the whole real line and 'rapidly decreasing' at $\pm\infty$, say the Schwartz space. In either case, the KdV equation has a unique solution $u(x, t)$ with prescribed initial value $u(x, 0) \in \mathscr{F}$: the equation can thus be pictured geometrically as defining a flow on $\mathscr{F}$. For either choice of $\mathscr{F}$ we have the following.

A1. The KdV equation has an infinite number of conserved quantities (first integrals), that is, functionals $I_q[u]$ that are constant on the trajectories of the KdV flow.

A2. In a sense to be clarified below, the KdV flow is the *Hamiltonian* flow corresponding to one of these integrals as Hamiltonian.

A3. The integrals I_q are all in involution in the sense of Hamiltonian mechanics, that is, the Poisson brackets $\{I_r, I_s\}$ all vanish.

Before explaining the meaning of these assertions, let me mention one other fact, perhaps the most basic one of all: the connexion of the KdV equation with the Schrödinger operator

$$L = -\partial^2 + u(x, t). \tag{2}$$

(We write $\partial \equiv \partial/\partial x$.) This is the deus ex machina from which everything else follows. The connexion first appeared in the remarkable paper [GD11], where it was observed that for u rapidly decreasing at infinity, the spectrum (or at any rate the discrete part of the spectrum) of the operator L does not change with time if u satisfies the KdV equation. Lax [GD12] then reasoned that this suggests that the operators $L(t)$ are all conjugate to a fixed one, say $L(t) = U(t)\,L(0)\,U(t)^{-1}$, or equivalently, $\partial_t(U^{-1}LU) = 0$. This last equation can be rewritten in the form

$$L_t = [P, L], \tag{3}$$

where $P = U_t U^{-1}$ (as usual, the square brackets denote the commutator $PL - LP$). Finally, Lax saw that the operator P in (3) can be taken to be a differential operator: if we set

$$P = -4\partial^3 + 6u\partial + 3u_x$$

it is easy to check that $[P, L]$ is (identically in u) an operator of order zero, and the zero-order term is the right hand side of equation (1). Equations (1) and (3) are therefore equivalent: (3) is now called the Lax representation of the KdV equation (1).

This story provides a good illustration of how a purely algebraic assertion (the equivalence of equations (1) and (3)) can have important consequences for the KdV flows on concrete function spaces (the time independence of the spectrum). Let me now elucidate the statements A1–A3 above. First, the conserved quantities. The underlying algebraic fact here is that there is an infinite sequence of identities

$$\partial_t H_q = \partial J_q \qquad (4)$$

that follow formally from (1). For example, we have

$$\partial_t(u^2) = 2uu_t = 2u(6uu_x - u_{xxx}) = \partial(4u^3 + u_x^2 - 2uu_{xx}).$$

(The first few H_q are listed in [GD], p.34; they are denoted there by R_q.) The 'conserved densities' H_q and 'fluxes' J_q are differential polynomials in u (that is, polynomials in u and its x-derivatives $u^{(i)}$). In either of our analytic contexts, we get the conserved quantities I_q in (A1) by integrating the H_q: $I_q[u] = \int H_q[u]\,dx$ (that explains the term 'conserved density'). The integrals are taken over a period, or over the whole line, depending on the choice of function space $\mathscr{F}$. In either case, it follows from (4) that if u satisfies the KdV equation, then

$$\partial_t I_q[u] = \int \partial_t H_q\,dx = \int \frac{\partial J_q}{\partial x}\,dx = 0,$$

so that I_q is indeed constant.

Notice that if we change H_q by adding on a term ∂f (f a differential polynomial), then the corresponding conserved quantity $\int H_q\,dx$ will be unchanged. Thus the conserved densities H_q should be regarded as well-defined only up to addition of a term ∂f. That explains why the lists of the H_q in [GD], p.34 and [DMN], p.61 do not agree exactly (even up to multiplication by constants).

Now the Hamiltonian formalism ((A2) above). The basic algebraic fact is that the KdV equation can be written in the form

$$u_t = \partial \frac{\delta H}{\delta u}, \qquad (5)$$

where $H = u^3 + \frac{1}{2}u_x^2$ is one of the conserved densities. Here $\delta H/\delta u$ is the formal variational derivative (see [GD], p.16, formula 10). In an analytic context we should write $\delta I/\delta u$ instead of $\delta H/\delta u$ (where as above $I = \int H dx$): this would be the usual Euler–Lagrange operator of the calculus of variations, applied to the functional $u \to \int H[u]\,dx$. Now, in that context it is standard that $\delta I/\delta u$ is analogous to the exterior derivative (or gradient) of a function; thus we think of (5) as analogous to Hamilton's equations

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix} \qquad (6)$$

on a finite dimensional phase space. (Vector notation: $q = (q_1, \ldots, q_n)$, etc.) The analogy of the skew differential operator ∂ in (5) with the skew matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in (6) is justified by a property of the operator ∂ ((iii) below) whose formulation involves the Poisson bracket referred to in (A3) above. I shall explain that first in the formal context. Let A be the algebra of all differential polynomials in u, with derivation $\partial = d/dx$ as in [GD], p.15, formula (4). For each $f \in A$, we define the 'vector field' $\partial_f \colon A \to A$ to be the unique derivation of A that commutes with ∂ and satisfies

$$\partial_f u = \partial \frac{\delta f}{\delta u} . \tag{7}$$

By the theorem in [GD], p.17, ∂_f depends only on the class of f in $A/\partial A$. We now define the Poisson bracket $(A/\partial A) \times (A/\partial A) \to A/\partial A$ by

$$\{f, g\} = \frac{\delta g}{\delta u} \, \partial \, \frac{\delta f}{\delta u} \pmod{\partial A}.$$

By integration by parts, it is easy to check that we have

(i) $\{f, g\} = -\{g, f\}$
(ii) $\{f, g\} = \partial_f g = -\partial_g f$,

as in ordinary Hamiltonian mechanics. The crucial property of the bracket, however, which is less easy to check, is the next one:

(iii) $\partial_{\{f,g\}} = [\partial_f, \partial_g]$.

(A closely related property is that the bracket satisfies the Jacobi identity.)

The intuitive meaning of all this becomes clear when we translate to one of the analytic contexts. (The reader should be warned, however, that there is some difficulty in doing this rigorously: see [33]). Thus now each $f \in A/\partial A$ defines a function $I_f[u] = \int f[u]\, dx$ on $\mathscr{F}$; the ∂_f in (7) is thought of as a vector field on the infinite dimensional manifold $\mathscr{F}$, and the Poisson bracket becomes

$$\{I_f, I_g\} = \int \frac{\delta I_g}{\delta u} \, \partial \, \frac{\delta I_f}{\delta u} \, dx$$

(integration corresponds to working modulo ∂A in the formal set-up). Thus we are in the usual situation of Hamiltonian mechanics, with the operator ∂ defining the 'symplectic structure' on the manifold $\mathscr{F}$. More accurately, ∂ defines what in finite dimensions is often called a 'Poisson structure' [23] or 'Hamiltonian structure' [38] on $\mathscr{F}$; that is, ∂ should be thought of as a skew form on the cotangent bundle (not the tangent bundle) of $\mathscr{F}$.

I shall not say any more about the formal properties of the KdV equation: the reader will find more details, as well as proofs of most of what I have said so far, in the article [GD]. Let me now discuss the articles [DMN] and [K], which are mainly devoted to the matters mentioned in (C) above. First we have to introduce the 'higher' KdV equations. These are the Hamiltonian flows obtained by taking the various conserved quantities I_q as Hamiltonians: that is, they are the equations of the form

$$u_t = \partial \frac{\delta H_q}{\delta u} \tag{8}$$

as H_q runs over the conserved densities of the KdV equation. Since the H_q are all in involution, the Hamiltonian formalism (more precisely, property (iii) above) shows that the corresponding vector fields ∂_t, and hence the corresponding 'higher KdV' flows, all commute with each other. In particular, the set of fixed points of any one of the higher KdV flows is invariant under all the others, amongst which is the flow of the KdV equation itself. Now, the fixed point set is given by setting the right hand side of (8) equal to zero; that gives an ordinary differential equation of the form

$$u^{(2q+1)} + f(u, u_x, \ldots, u^{(2q-1)}) = 0,$$

where f is a polynomial. The solutions of this form a finite dimensional space V^{2q+1}. We have the following.

C1. The space V^{2q+1} is foliated by $2q$-dimensional symplectic manifolds M^{2q} on each of which the KdV flow is a completely integrable Hamiltonian system in the classical sense (see, for example, section 6 in the article of Vinogradov and Kupershmidt).

C2. More interesting still, the KdV equation with an initial value $u(x, 0) \in V^{2q+1}$ can be integrated explicitly (generally in terms of the Riemann θ-function of a hyperelliptic Riemann surface).

I shall discuss (C1) only very briefly (for details see Ch.3 of [GD] and the papers [DMN64] and [9]: summaries of [DMN64] can be found in the concluding remarks to [DMN] and in the appendix 1 to [K]). Let me just explain what the manifolds M^{2q} are: they are the spaces of solutions of the equations

$$\frac{\delta H_q}{\delta u} = c \tag{9}$$

for various constants c; clearly, as c varies we get exactly the solutions of the stationary equation (8). Equation (9) can be written in the strictly Lagrangian form

$$\frac{\delta}{\delta u}(H_q - cu) = 0;$$

the symplectic structure on the space of solutions M^{2q} is now obtained by a procedure that is standard in the calculus of variations, and is explained in a formal context in [GD], ch.1.

The explicit integration of these restricted KdV flows involves some algebraic geometry: the paper [11] tries to persuade us that this arises inevitably if we try to implement Liouville's procedure for integrating an integrable Hamiltonian system. However, the papers [DMN] and [K] use a more direct approach based on the Lax representation (3) of the KdV equation. We need the corresponding description of the higher KdV equations. Let L as before be the Schrödinger operator (2). It is not hard to see that for each integer q there is a unique differential operator of the form

$$P_q = \partial^{2q+1} + (\text{lower order terms})$$

whose coefficients are differential polynomials in u with zero constant term, such that $[P_q, L]$ is an operator of order zero. The equations

$$L_t = [P_q, L]$$

are the higher KdV equations. (A proof that this construction of the higher KdV equations agrees with our earlier, Hamiltonian, one is given in [GD], p.42.) The higher stationary equations thus take the form $[P_q, L] = 0$; more generally, we can consider equations of the form

$$[Q, L] = 0, \tag{10}$$

where Q is some linear combination of the P_q. We want to find solutions $u(x, t)$ of the KdV equation which for each fixed value of t satisfy the ordinary differential equation (10). It is possible to discuss this directly, as is done in [K], but the ideas will perhaps be more transparent if we first study equation (10) by itself: it turns out that once we have a good description of the space of solutions of (10), it is a simple matter to describe also the KdV flow on this space. The main point is that the solutions of (10) can be constructed from certain algebro-geometric data; however, to understand the motivation for the construction it is best first to see how to go in the other direction. Suppose then that we have a solution $u(x)$ (possibly complex valued) of (10); it gives us a pair of commuting differential operators Q, L. The corresponding algebro-geometric data consists essentially of the joint eigenvalues (spectrum) and eigenfunctions of these operators. In more detail, let V_λ denote the λ-eigenspace of L; thus V_λ is a two-dimensional complex vector space. Since Q commutes with L, it preserves each space V_λ: let $f(\lambda, \mu)$ denote the characteristic polynomial of Q acting on V_λ:

$$f(\lambda, \mu) = \det (Q \mid V_\lambda - \mu.Id).$$

The joint spectrum S of L and Q is clearly given by the equation $f(\lambda, \mu) = 0$. We have the following facts.

(i) f is a polynomial in two variables, so that the spectrum $S \subset \mathbf{C}^2$ is an affine algebraic curve.

(ii) Near infinity, S looks like the curve $\mu^2 = -\lambda^{2q+1}$, (where $2q + 1$ is the order of Q); thus S has just one point at infinity. Let $\bar{S}$ denote the compact curve obtained by adjoining the non-singular point ∞ to S.

(iii) Near the point $\infty \in \bar{S}$, we can take as local parameter k^{-1}, where $k^2 = -\lambda$.

(iv) At each non-singular point $z = (\lambda, \mu)$ of S, the corresponding joint eigen-space of L and Q is one-dimensional.

For simplicity (as in [K]), let us suppose that S is non-singular; then $\bar{S}$ is a compact Riemann surface. For each $x \in \mathbf{C}$, let $D(x) \subset S$ denote the divisor of points $z \in S$ such that the corresponding joint eigenfunction vanishes at x (we assume that $u(x)$, and hence the coefficients of the operators L and Q, are defined, and have no singularities, at least for x in some neighbourhood of the origin in $\mathbf{C}$). Let $\psi(z, x)$ ($z \in S$, $x \in \mathbf{C}$) denote the joint eigenfunction, normalized so that $\psi(z, 0) = 1$. Naturally, this introduces a pole if $z \in D(0)$.

(v) The divisors $D(x)$ have degree g = genus of $\bar{S}$, and are non-special. (Non-special means that there is no non-constant meromorphic function on $\bar{S}$ whose divisor of poles is $D(x)$ (or less). This property is a little harder to see than the others.)

(vi) The function $\psi(z, x)$ has the properties

(a) Away from ∞, ψ is analytic except for poles when $z \in D = D(0)$ (independent of x).

(b) Near ∞, $\psi \sim \exp(kx)$, where k is as in (iii).

The crucial point now is that this construction can be run backwards: suppose we are given a compact Riemann surface $\bar{S}$ of genus g, a point $\infty \in \bar{S}$, a non-special positive divisor D of degree g (not involving the point ∞) and a local parameter k^{-1} near ∞. Then there is a unique function $\psi(z, x)$, $z \in \bar{S}$, $x \in$ (open neighbourhood of 0 in $\mathbf{C}$), with the properties (a) and (b) in (vi). Furthermore, we can reconstruct (uniquely) the differential operators of which ψ is the joint eigenfunction: let λ be any meromorphic function on $\bar{S}$ whose only singularity is a pole at ∞. Then there is a unique ordinary differential operator L_λ in $\partial/\partial x$ such that

$$L_\lambda \psi(z, x) = \lambda(z)\, \psi(z, x) \text{ for all } z \in \bar{S} - \{\infty\}.$$

It is clear that the operators L_λ for various λ all commute with each other. The order of L_λ is equal to the order of the pole of λ at ∞. Thus if we want (as we do) one of the operators L_λ to have order 2, we must choose $\bar{S}$ so that there is a function λ on $\bar{S}$ whose only singularity is a double pole at ∞: that is, $\bar{S}$ must be hyperelliptic (and ∞ must be a 'Weierstrass point').

This construction is well explained in [K], so let me just comment on one point that Krichever passes over, the existence and uniqueness of the function ψ. That is most easily seen as follows. Suppose we replace (b) in (vi) by (b′): near ∞, $\psi \sim$ (const). $\exp(kx)$. Then conditions (a) and (b′) say that, for fixed

x, ψ is a section of the holomorphic line bundle $\{D\} \otimes \{e^{kx}\}$ on $\bar{S}$. Here $\{D\}$ is the usual bundle associated with the divisor D, and $\{e^{kx}\}$ denotes the bundle defined by the single transition function e^{kx} in a punctured neighbourhood of ∞. Now, if x is near 0, this bundle is near to $\{D\}$, and hence inherits the non-special character of $\{D\}$; that is, its space of sections is one-dimensional. Hence fixing the value of the constant in (b′) gives us a unique function ψ. This argument also shows that as x varies the corresponding bundle (which can also be described as the bundle defined by the divisor $D(x)$) moves along a straight line in the Jacobian of S (which is a g-dimensional complex torus). That follows at once from the formula $e^{kx}e^{ky} = e^{k(x+y)}$, since the group operation in the Jacobian (tensor product of line bundles) corresponds to multiplication of transition functions.

So far we have discussed only the stationary equation (10); but it is now easy to describe the KdV flow on the space of solutions of (10). (Indeed, it is equally easy to describe the solutions of equations of 'Kadomtsev–Petviashvili' type, involving an extra variable y: this is taken as the starting point in [K].) Suppose we have a solution $u(x, t)$ of the KdV equation which for each t also satisfies (10): thus we have everything we have said so far for each value of t. Equation (3) implies that the joint spectrum S of L and Q is independent of t; and if we normalize the joint eigenfunction $\psi(z, x, t)$ by $\psi(z, 0, 0) = 1$, $\psi_t = P\psi$ (where P is the operator in the Lax representation (3) of the KdV equation), then it is not hard to see that its asymptotics at ∞ will be given by $\psi \sim \exp(kx - 4k^3 t)$. (The k^3 reflects the fact that the operator P in (3) has order 3.) By the same argument as before it follows that as t changes the bundle defined by the divisor $D(x, t)$ of zeros of ψ moves along a straight line in the Jacobian (of course a different one from before). I refer to [K] for the details of the inverse construction, and also for the explanation of how to translate this geometrical description of the flow into an explicit formula for $u(x, t)$: the point is that the function ψ, being characterized by its poles and asymptotics at ∞, can be written down explicitly in terms of the θ-function of $\bar{S}$.

It remains to say something about the 'finite-zone' periodic solutions of the KdV equation, which play a large role in the article [DMN]. Suppose $u(x)$ is a (now real) C^∞ periodic function with period T; let $\hat{T}$ denote the operator of translation over a period: $\hat{T}\varphi(x) = \varphi(x + T)$, and as usual let $L = -\partial^2 + u$. Clearly L and $\hat{T}$ commute, so we can again consider their joint spectrum S, the space of 'Floquet multipliers'. This is again a Riemann surface, but in general of infinite genus: its branch points lie over the simple periodic and anti-periodic spectrum of L, that is, over the points λ such that $\hat{T}$ acting on the λ-eigenspace of L is not diagonalizable. By definition, the finite-zone potentials $u(x)$ are those for which there is only a finite number of such points, so that the Riemann surface S has finite genus. The KdV flow on the space of such functions $u(x)$ can be described exactly as before, as a straight line flow on the Jacobian of $\bar{S} = S \cup \{\infty\}$. In fact a periodic function $u(x)$ is finite-zone precisely when it is a solution of one of the higher stationary equations (10) (see [DMN], p.78 and p.86); the Riemann surface S can be described as the joint spectrum either of L and $\hat{T}$, or of L and Q. Thus the finite-zone periodic theory is really the intersection of two theories: (i) the algebro-geometric theory of Krichever that I described above (ii) the general periodic theory.

In this volume we are not really concerned with the integration of the KdV equation on the infinite dimensional spaces mentioned in (B) at the beginning of the introduction; so let me just say a very few words about it. I already mentioned above the main idea for the general periodic problem: it proceeds just like the finite-zone case, but one has to work with Riemann surfaces of infinite genus, which naturally causes some problems. Details can be found in [28]. The solution of the initial value problem in the class of functions rapidly decreasing at infinity is provided by the famous 'inverse scattering method': again the idea is that as t changes the spectrum of L stays constant and we study the change in the eigenfunctions: but now 'spectrum' means the L^2 spectrum, and the evolution of the eigenfunctions is followed by means of the 'scattering data'. An introduction to this theory is given in [DMN], p.71–73, and a rigorous account can be found in [7].

We saw above that the finite-zone periodic theory could be viewed as the intersection of the algebro-geometric and general periodic theories. It is natural to ask what is the corresponding intersection with the rapidly-decreasing-at-infinity theory, that is, what are the solutions $u(x, t)$ of the KdV equation that are rapidly decreasing at infinity, and satisfy one of the higher stationary equations (10) for each fixed value of t. It turns out that these are precisely the 'n-soliton' solutions, consisting of n travelling waves that appear to pass through each other and emerge unscathed from the non-linear interaction. From the algebro-geometric point of view, these solutions arise in the case when the joint spectrum of L and Q is a (singular) rational curve with n ordinary double points. Unfortunately, I do not know of anywhere in the literature where that is explained in much detail.

Finally, a few remarks about the references that follow. These by no means represent a serious attempt to bring the bibliography up to date, but are simply a selection based on my own prejudices; furthermore, the list is confined to papers that seem to me to be a direct continuation of lines of thought to be found in the articles reprinted here. Many of the papers are concerned with the problem of generalizing the theory to the case of Lax equations $L_t = [P, L]$ in which L is an operator of higher order (and possibly with matrix coefficients). Some of the physically interesting equations obtained in this way are listed in [DMN], ch.1. Indeed, the article [K] is written at this level of generality: the generalization of the algebro-geometric theory is straightforward. The formal side of the theory presents more difficulty; in particular, the Hamiltonian formalism is more complicated than in the case of the KdV equation. An introduction to this is given in [K], appendix 1; details can be found in the papers

[1, 10, 12, 13, 21, 22, 25, 40, 41, 42, K32, K33]. The papers [16, 18, 19, 26, 31, 39] discuss the 'higher rank' algebro-geometric solutions of Lax equations; these are the ones that arise when the joint eigenspaces of the commuting operators Q, L, are not one-dimensional. (That can happen only when the orders of Q and L are not relatively prime, which is why the KdV equation has no such solutions.) The preliminary account of this theory given in [K], section 2.3 contains some errors, which are corrected in [16]. The papers [1, 14, 22, 34, 35, 36, 37, 38] forge a connexion between this subject and Lie theory; let me just mention the result of Adler [1] and Lebedev–Manin [22] that the Hamiltonian structure of the KdV equation (and also of the more general Lax equations) can be interpreted as the standard Kirillov–Kostant structure on the dual of a certain Lie algebra of 'formal integral operators'. The papers [3, 6, 40] are those referred to as preprints in [K], pages 161 and 163; the work of Krichever mentioned in [K], p.164 is published in [15, 17]. To end, let me recommend the two survey articles [25, 27].

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