ON THE ABSTRACT GROUP OF AUTOMORPHISMS

László Babai
Dept. Algebra, Eötvös University,
Budapest, Hungary.

ABSTRACT

We survey results about graphs with a prescribed abstract group of automorphisms. A graph X is said to ***represent*** a group G if Aut X ≅ G. A class C of graphs is (f)-***universal*** if its (finite) members represent all (finite) groups. Universality results prove independence of the group structure of Aut X and of combinatorial properties of X whereas non-universality results establish links between them. We briefly survey universality results and techniques and discuss some non-universality results in detail. Further topics include the minimum order of graphs representing a given group (upper vs. lower bounds, the same dilemma), vertex transitive and regular representation, endomorphism monoids. Attention is given to certain particular classes of graphs (subcontraction closed classes, trivalent graphs, strongly regular graphs) as well as to other combinatorial structures (Steiner triple systems, lattices). Other areas related to graph automorphisms are briefly mentioned. Numerous unsolved problems and conjectures are proposed.

0. AUTOMORPHISM GROUPS - A BRIEF SURVEY

In two of his papers in 1878, Cayley introduced what has since become familiar under the name "Cayley diagrams": a graphic representation of groups. Combined with a symmetrical embedding of the diagram on a suitable surface, this representation has turned out to be a powerful tool in the search for generators and relators for several classes of finite and finitely generated groups. This approach is extensively used in the classic book of Coxeter and Moser [CM 57] where a very accurate account of early and more recent references is also given.

Automorphism groups of combinatorial objects with high symmetry (projective spaces, block designs, more recently strongly regular graphs, two-graphs, generalized polygons) have always played an important role in group theory (cf. [Bi 71], [Ka 75], [ST 81], [Ti 74]). A study of graphs satisfying certain strong symmetry conditions was initiated, well ahead of time, by Tutte's fundamental paper on s-transitive trivalent graphs [Tu 47]. A graph is s-transitive if its automorphism group is transitive on arcs of length s. The classification of finite simple groups, which rumour says has recently been completed, has a substantial effect on the study of highly symmetrical graphs. One example is the result of R. Weiss [We 81] who has shown that the bound $s \leq 7$ holds for s-transitive graphs of arbitrary valence ≥ 3 assuming the known list of 2-transitive permutation groups is complete. (The completeness of this list follows from the classification of finite simple groups [CKS 76], cf. [Ca 81].)

Another class of graphs with a high degree of symmetry are rank-3 graphs and more generally distance-transitive graphs. These are graphs whose automorphism group acts transitively on the set of pairs of vertices at any fixed distance. It is this class of graphs to which, ever since the beautiful paper of Hoffman and Singleton [HS 60], methods of linear algebra have most successfully been applied. The method is to derive regularity conditions (in terms of numerical parameters of the graphs) from the symmetry conditions and subsequently to translate the combinatorial regularity into information on the eigenvalues of associated matrices. A detailed exposition of some of the highlights of this theory can be found in N. Biggs' excellent book [Bi 74]. More recent results involving such methods include the Cameron-Gol'fand theorem which describes all 5-homogeneous graphs ([Ca 80], [Gnd 78]; cf. [Sm 75] and [CGS 78]). (A graph is k-homogeneous if any isomorphism between induced subgraphs on at most k vertices extends to an automorphism of the graph.) Cameron has extended this result to distance-regular graphs (under the condition of metrical 6-homogeneity) [Ca 80], cf. [Ca x]. In fact the actual results are much stronger inasmuch as they assume only k-regularity, a combinatorial condition which appears to be much weaker than k-homogeneity. (For $k = 2$, k-regular graphs are strongly regular; metrically 2-regular graphs are distance regular. These conditions do not imply the presence of any non-identity automorphism.) Gol'fand has now (privately) announced the classification of all ultra-homogeneous association schemes. (Ultrahomogeneous means k-homogeneous for all k.)

Colored, directed graphs are a natural object on which uniprimitive (primitive but not doubly transitive) groups act. The colors correspond to the orbits on pairs of elements. A nice introduction to the ideas derived from this representation is given in [Ne 77]. Besides powerful matrix methods ([FH 64], cf. [Hi 75a,b]) which are a common generalization of the eigenvalue techniques used for association schemes [BS 52], [De 73] and of group representation theory, there still seems to be a lot of room left for elementary graph theoretic considerations. The colored digraphs satisfying certain regularity conditions implied by (but, of course, not equivalent to) primitive group action are called primitive coherent configurations. A combination of inequalities involving valence and diameter of constituent digraphs (color classes) of such configurations have led to substantial progress on a classical problem in group theory [Ba 81]: *If* G *is a unipriprimitive permutation group of degree* n *then* $|G| < \exp(4\sqrt{n}\log^2 n)$. (This result is sharp up to a factor of 4logn in the exponent. The best previous bound was Wielandt's $|G| < 4^n$ [Wi 69].)

The results indicated in the above paragraphs have dealt with graphs satisfying regularity rather than symmetry conditions and are therefore not affected by progress in group theory. If however we assume our graphs have a high degree of symmetry (in terms of automorphisms), the classification of finite simple groups becomes relevant. Cameron [Ca 81] mentions that J. Buczak has determined all 4-homogeneous graphs, making a strong use of the list of finite simple groups. Cameron himself [Ca 81] has determined all primitive permutation groups of degree n and of order greater than $n^{(1+\varepsilon)\log n}$ ($\varepsilon \to 0$ while $n \to \infty$)(assuming the classification). It turns out that all such groups act as subgroups of S_k wr S_m on the set of ordered m-tuples of t-subsets of a k-set where $n = \binom{k}{t}^m$. One can naturally define an association scheme on this set (a combination of the Hamming and Johnson schemes). Our "large" primitive group acts on such a scheme. Such results may motivate us to ask whether these are the only association schemes with large automorphism groups. In particular, *is it true for some constant* C *that if* $|\mathrm{Aut}\ X| > n^{C\log n}$ *for a strongly regular graph* X *then* X *is either complete multipartite, or the line graph of a complete or of a complete bipartite graph, or the complement of such a graph*?

Weaker conditions of symmetry such as a vertex-transitive automorphism group have interesting consequences on the structure of the graph (cf. Problems 13-19 in [Lo 79, Ch 12]). Although only four non-Hamiltonian connected vertex-transitive graphs are known, I don't feel tempted to suggest that there are only finitely many of them. It seems rather that a lack of good methods to prove non-Hamiltonicity hinders construction of an infinity of examples.

An interesting class of vertex-transitive graphs are provided by Cayley graphs. Since Maschke's paper [Ma 1896], continuing attention has been given to their embeddings on surfaces ([CM 57], [Wh 73]), but little has been done to explore other graph theoretic properties of Cayley graphs. A study of subgraphs and of the chromatic number of Cayley graphs has been initiated by [Ba 78a,b]. It is an open question whether *there is a constant* C *such that the chromatic number of any Cayley graph of a finite group with respect to an irredundant set of generators is less than* C.

Cayley diagrams play an essential role in constructing graphs with a prescribed automorphism group. The basic question, which groups are isomorphic to the automorphism group of a graph, was raised by König [Kö 36, p.5]. One can ask the same question for all kinds of mathematical structures in the place of graphs. This is what B. Jónsson [Jo 72] calls the ***abstract representation problem.*** Several significant directions of research on this problem find their roots in the work of R. Frucht. First, Frucht proved that ***every finite group is isomorphic to the automorphism group of some finite graph*** [Fr 38]. Then he went on proving that one can even require these graphs to be ***trivalent*** [Fr 49]. In another paper [Fr 50] he gave examples how to use graphs to construct ***lattices*** with given automorphism groups. Furthermore, in all these papers he was very much concerned ***about the size of the graphs*** he had constructed. There is one more classical paper on the subject: Birkhoff proved that every group is represented as the (abstract) group of automorphisms of a ***distributive lattice*** [Bi 45]. These results were followed by countless further ***universality results***: constructions which prove that every group is isomorphic to the automorphism group of some object in a given class. Characteristic for the proofs is that the structure of the groups plays little role. The extent to which group structure could be ignored was forcefully demonstrated by results of Pultr, Hedrlin and their colleagues who found that most results generalize to ***endomorphism semigroups*** (even to categories).

Although there is a large number of papers proving independence of the abstract group Aut X and various properties of the object X (graph, ring, lattice, etc.), little has been done to establish links between properties of X and the structure of Aut X.

The aim of the present note is to survey some aspects of the abstract representation problem, with an emphasis on the (missing?) link. By presenting a large collection of open problems, we try to draw attention to the other side of a coin, one side of which has already been studied in great detail.

Basic definitions will be given in Section 1. We give a brief survey of some interesting universality results in Section 2. Contraction-closed classes of graphs are investigated in Section 3. Such classes generalize the notion of graphs embeddable on a given surface. We give a detailed proof of the fact that most classical simple groups are not represented as the (abstract) automorphism group of any graph in such a class. This is an example of the non-universality results we seek. The proof involves a study of contractions of certain Cayley graphs. The "Subcontraction conjecture" 3.3 is hoped to lead to a deeper understanding of the graph structure forced by group action. Non-universal classes of lattices are the subject of Section 4, where attempts to link combinatorial parameters such as order dimension to the automorphism group are described. Section 5 is devoted to the problem of minimizing the size of graphs representing a given group. In order to illustrate the methods available, we outline the proofs of both upper and lower bounds. Clearly, refinement of universality proofs yield upper bounds. Correspondingly, lower bounds are scarce. The "Edge-orbit Conjecture" 5.21 may indicate the possibility of an interesting lower bound. Related results and problems on lattices conclude Section 5.

Vertex-transitive representations of a given group are the subject of Section 6. The graphical and digraphical regular representation problems belong to this section. In contrast to Section 5 which is dominated by open problems, in this section we are able to survey many fine results.

Finally in Section 7 we consider the extension of some of the problems treated to endormophism monoids. We outline the proof of the basic representation theorem, keeping the minimizing problem of the number of vertices in mind. After a brief statement of some of the important universality results we turn our attention once again to what we feel ought to be an organic part of the theory but is almost entirely missing, links between the structure of the graph X and of its (abstract) endomorphism monoid.

The reader wishing to obtain a broader view of the area of graph automorphisms is referred to the excellent survey by P. Cameron [Ca x]. With that paper being in print, I have elected to write on this more compact subject which presents many problems of combinatorial rather than group theoretic nature. From among those areas closely related to graph automorphisms but omitted from this introduction let me mention the algorithmic complexity of graph isomorphism (cf. [Ba 80c]). The recent breakthrough by E.M. Luks [Lu 80] signals the relevance of the structure of the automorphism group to a depth where even the classification of finite simple groups may become relevant.

1. DEFINITIONS, NOTATION

A digraph is a pair $X = (V,E)$ where $V = V(X)$ is the set of vertices and $E(X) = E \subseteq V \times V$ is the set of directed edges. A graph is a digraph with $E = E^{-1}$ and with no loops. K_n stands for the complete graph on n vertices. A ***vertex-coloured*** (di)graph is a (di)graph X together with a function $f : V \to K$ which maps V into some sets of colors. (f is not a good coloring in the sense of chromatic graph theory.) An ***edge-coloured*** digraph is a set V together with a family of binary relations $E_1,\ldots,E_m \subseteq V \times V$. ($1,\ldots,m$ are the edge-colors.)

An automorphism is an $X \to X$ isomorphism. Automorphisms preserve colors by definition.

Let G be a group and H a set of generators for G. We define the ***Cayley diagram*** $\Delta(G,H)$ to be an edge-colored digraph with vertex set G. Colors are members of H. There is an edge of color $h \notin H$ joining g to gh for every $g \in G$.

The ***Cayley digraph*** $\Gamma(G,H)$ has the same set of vertices and edges but no colors. (Therefore, it may have more automorphisms than $\Delta(G,H)$.) $\Gamma(G,H)$ is a ***Cayley graph*** if $H = H^{-1}$ and $1 \notin H$. Cayley graphs are connected. Finite Cayley digraphs are strongly connected.

Throughout the paper, G denotes a group, n the order of G and d the minimum number of generators of G.

Both the beginning and the end of proofs are indicated by □.

2. PRESCRIBING THE ABSTRACT GROUP

In this section we consider the following type of problem: Given a group G find a graph X (or a block design, a lattice, a ring, etc.) such that the automorphism group Aut X is ***isomorphic*** to G. Such an object X will be said to ***represent*** the group G. A class $\mathcal{C}$ of objects is said to represent a class $\mathcal{G}$ of groups if, given $G \in \mathcal{G}$ there exists $X \in \mathcal{C}$ such that Aut $X \cong G$. We call $\mathcal{C}$ ***universal***, if every group is represented by $\mathcal{C}$. We say that $\mathcal{C}$ is f-universal if every ***finite*** group occurs among the groups represented by finite members of $\mathcal{C}$.

The first natural question was put by König in his classic monograph [Kö 36, p.5]: Which groups are represented by graphs? Frucht has soon settled the question for the finite case, proving that (the class of) graphs is f-universal. In other words:

THEOREM 2.1 [Fr 38] ***Given a finite group*** G ***there exists a finite graph*** X ***such that*** Aut $X \cong G$.

Frucht's proof has since become standard textbook material [O 62], [Ha 69], [Lo 79], [Bo 79]. The idea is (i) to observe that the automorphism group of the colored directed Cayley diagram of G w.r. to any set of generators is isomorphic to G; (ii) to get rid of colors and orientation by replacing colored arrows by appropriate small asymmetric (automorphism free) gadgets. The same trick applies to infinite groups. We need many asymmetric graphs for that; they can be obtained from a well-ordered set by adding Frucht type gadgets between any pair $a < b$. We conclude that Frucht's theorem extends to the infinite case:

THEOREM 2.2 ***Graphs are universal.***
(See [Bi 45], [Gr 59], [Sa 60].)

The next problem that arises is to find subclasses of graphs and classes of other (combinatorial, algebraic, topological) objects that are universal. This direction was again initiated by Frucht's fundamental discovery:

THEOREM 2.3 [Fr 49]. *Trivalent graphs are f-universal.*

Although, as Professor Frucht informs me, M. Milgram has found some gap in the original proof, there are several proofs available now (see e.g. [Lo 79, Ch. 12, Problem 8] or Section 5 of this paper). [Fr 49] had a great impact; its merit was not just proving a theorem but giving a new insight - an accomplishment offered by few flawless papers.

Another fundamental result of this kind was published a few years earlier in Spanish by G. Birkhoff:

THEOREM 2.4 [Bi 45]. *Distributive lattices are universal.*

These surprising results already foreshadow the onesidedness of later development. Take almost any interesting class of combinatorial or algebraic structures; this class is universal. (There are easy exceptions: trees, for instance. Automorphism groups of finite trees have been characterized by Pólya [Pó 37] as repeated direct and wreath products of symmetric groups. The idea goes back to Jordan [J 1869] who counted tree automorphisms. As for infinite trees, it is easy to see that if a finite group is represented by an infinite tree then it is represented by a finite one as well. - Groups are not universal either: it is easy to see that for no group G ($|G| \geq 3$) is Aut G a cyclic group of odd order.)

Some of the universality results are straightforward. Bipartite graphs are an example. Just take a connected graph X which is not a cycle, and halve each edge by inserting a new vertex of degree two. The obtained graph Y is bipartite and satisfies Aut Y $\cong$ Aut X. Now, use Theorem 2.2 to prove that bipartite graphs are universal.

It is quite easy to prove that Hamiltonian, k-connected or k-chromatic graphs are f-universal; it is somewhat more difficult to extend Frucht's Theorem 2.3 to regular k-valent graphs ($k \geq 3$). Sabidussi's paper [Sa 57] proving that these and some other classes of graphs were f-universal, was soon considered as compelling evidence to support the view that "*requiring* X *to have a given abstract group of automorphisms was not a severe restriction*" [Ha 69, p. 170], [O 62, Ch 15.3]. Izbicki proved that certain combinations of Sabidussi's conditions are still insufficient to restrict the automorphism group [Iz 57, 60]. Universality results in algebra and topology were inspired by de Groot's papers [Gr 58, 59]; one of his results there is that commutative rings are universal. A surprisingly strong version of this was given by Fried and his undergraduaate student Kollár:

THEOREM 2.5. [FK79,81] *Fields are universal*

Finite extensions of $\mathbb{Q}$ are universal over finite groups. (We have to note here that these extensions are not normal. Noether's classical question whether every finite group is the Galois group of a polynomial over $\mathbb{Q}$ remains open, see [Sh 54].)

From the numerous universality results for finite combinatorial structures, let me quote some appealing ones. They are due to E.Mendelsohn:

THEOREM 2.6 [Me 78a]. *Steiner triple systems as well as Steiner quadruple systems are* f-*universal.*

COROLLARY 2.7 [Me 78b]. *Strongly regular graphs are* f-*universal.*

In order to see how 2.7 follows from 2.6, let X be a Steiner triple system. Take its line graph L(X) (vertices are triples from X, adjacency means non-empty intersection). Points of X correspond to maximum cliques in L(X). Conversely, every maximum clique of more than 7 vertices in L(X) corresponds to a point in X. (This follows, for instance, from Deza's theorem [De 74] (see [Lo 79, Ch. 13 Probl. 17]): if the intersection of every pair of more than n^2-n+1 n-sets are of the same size, then these pairwise intersections coincide.) We conclude that for $|V(X)| > 15$, one can recover X from L(X) hence Aut X $\cong$ Aut L(X) proving 2.7.

The proofs of 2.5, 2.6 and many other similar results start from a graph X with given automorphism group, and build an appropriate object X' such that Aut X $\cong$ Aut X'. It is usually easy to find X' such that Aut X $\lesssim$ Aut X'. The task is then to make it sure that X' has no superfluous automorphisms. There are interesting cases, however, where even the "subgroup problem" is open.

PROBLEM 2.7. *Prove for every* $k \geq 3$, *that, given a finite group* G, *there is a* BIBD *of block size* k (a 2-(v,k,1)-design) X *such that* G $\lesssim$ Aut X.

Such X is easily found if $k = p^\alpha$ or $p^\alpha+1$ (p prime): take affine or projective spaces of high dimension over $GF(p^\alpha)$ with the lines as blocks. For such k one can in fact prove that BIBD's with block size k are f-universal [Ba y], extending the STS result of E. Mendelsohn.

CONJECTURE 2.8. BIBD's *of block size* k *are* f-*universal for any fixed* $k \geq 3$.

We note that an affirmative answer to 2.7 is known when k is a multiple of the order of G [Wil].

Although there are many more interesting universality results and some open problems of this kind, the literature on them has grown out of proportion without the healthy balance of theorems that would provide links between the group structure of Aut X and the combinatorial nature of X. This situation may derive largely from the nature of the subject but to some extent also from the pressure to publish or perish. Techniques for constructing objects with a given automorphism group are well developed, and such (sometimes easy, sometimes ingenious, often tedious) constructions dominate the subject. There is a chronic lack of questions pointing to possible links rather than to independence of the structure of Aut X and properties of X. My main objective in the next two sections is to show that such links do exist and exploring them could be a worthwhile task.

3. NON-UNIVERSAL CLASSES OF GRAPHS

Tournaments are not universal. Their automorphism groups have odd order for the simple reason that any involution would reverse an edge. On the other hand, every finite group of odd order can be represented by a tournament [Mo 64] (cf. [Lo 79, Ch. 12, Problem 7]). So this again is a universality type result rather than the kind we seek.

Turán asked in 1969 whether planar graphs were f-universal. The negative answer [Ba 72] was the starting point of the author's research in this direction. Graphs embeddable on a given compact surface were shown to be non-universal [Ba 73] and the automorphism groups of planar graphs have been fully described in terms of repeated application of a generalization of wreath product, starting from symmetric, cyclic, dihedral groups and the symmetry groups of Platonic solids (A_4, S_4, A_5) [Ba 75]. I expect that in some sense, such structure theorems should hold under much more general circumstances. As a first step, we find the following non-universality result.

Contraction of a graph X onto a graph Y is a map $f : V(X) \to V(Y)$ such that (i) $u,v \in V(Y)$ are adjacent iff $u = f(x)$, $v = f(y)$ for some adjacent pair x, y of vertices of X; (ii) the subgraph of X induced by $f^{-1}(u)$ is connected for every vertex u of Y.

Y is a *subcontraction* of X if Y is a subgraph of a contraction of X. Clearly, the class of finite graphs embeddable on a given surface is *subcontraction closed*. It is also easy to see that there are subcontraction closed classes of finite graphs, not embeddable on any compact surface. (Take the graphs with no block of more than 5 vertices, for instance.) The interest in subcontraction closed classes stems among other things from Hadwiger's conjecture. For the graph theory of subcontraction we refer to [Ma 68], [Ma 72], [O 67].

The principal result intended to illustrate our point is this.

THEOREM 3.1. [Ba 74a]. *If a subcontraction closed class of graphs is f-universal then it contains all finite graphs.*

The infinite version of this is still open:

CONJECTURE 3.2. *If a subcontraction closed class of graphs is universal, then it contains all graphs.*

An equivalent formulation of this conjecture is the following: *Given a cardinal* κ *prove that there is a group* G *such that for any graph* X, *if* Aut $\cong$ G *then* X *contains a subdivision of the complete graph* K_κ.

A good candidate for such G might be a large alternating group or some other simple torsion group.

Non-universality results immediately call for an investigation of the structure of those groups actually represented by the class of objects in question. As a first step toward a structure theory of these groups, one might like to find out which *simple* groups are represented. My favourite problem pertains to this question.

SUBCONTRACTION CONJECTURE [Ba 75] 3.3. *Let C be a subcontraction closed class of graphs, not containing all finite graphs. Then the set of non-cyclic finite simple groups represented by C is finite.*

Another way of stating this problem is this:

CONJECTURE 3.3'. *Given an integer* k *find an* N = N(k) *such that if* G *is a finite simple group of composite order greater than* N(k) *and* X *is a graph such that* Aut X $\cong$ G *then* X *has a subcontraction to the complete graph* K_k.

There are partial results in this direction. In [Ba 74a,b] it is shown that the conclusion of 3.3' holds if G contains $Z_p \times Z_p \times Z_p$ (the elementary abelian group of order p^3) for some large prime. (We assume G is simple.) Another large class of finite simple groups G is taken care of by the following result. Let p and r be prime numbers, $p \equiv 1 \bmod r$ and let H(p,r) denote the nonabelian group of order pr.

THEOREM 3.4. *For every* k *there is an* M = M(k) *such that if the finite simple group* G *contains* H(p,r), r > M(k) *and* Aut X $\cong$ G *then* X *has a subcontraction onto* K_k.

□We sketch the proof which goes along the lines of [Ba 74a]. We continue the present discussion after Remark 3.11 so the reader wishing to omit proofs may turn to that page. First one proves the following lemma.

LEMMA 3.5. *If* Aut X $\cong$ G *is a finite simple group then* X *has a subcontraction* Y *such that* (i) Y *is connected,* (ii) G *acts as a subgroup of* Y, (iii) *this action is transitive on edges and* (iv) *no vertex of* Y *is fixed under the action of* G.

(A minimal subcontraction of X satisfying (i), (ii) and (iv) will also satisfy (iii).)

If Aut Y is transitive then Y is regular. Otherwise Aut Y has two orbits, Y is bipartite and semiregular (vertices in each color class have equal valences). If all vertices of Y have large valence, a theorem of Mader yields the result: