INTRODUCTION

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Professor Arnold is well known for his researches on a variety of topics in pure and applied mathematics, but perhaps no field owes more to him than singularity theory. In this volume are collected 7 survey articles of his on singularity theory that have appeared over the last decade. The first of these, written at a time (1968) when the subject was rapidly opening up, remains an excellent general introduction to the field as a whole.

However the core of the volume, consisting of 3 articles which appeared in 1973–75, consists of an account of the classification of critical points of smooth functions, and of the reinterpretations of a key class of functions (those with normal form depending on at most one parameter) – in relation to Lie groups, spherical and hyperbolic triangles, and definiteness of the intersection form – obtained by Arnold and his students during that period. Together, these results constitute one of the most beautiful discoveries in mathematics in recent years: and the further detailed study of these classes of singularities has revealed at each stage unexpected and rich structure.

Although Arnold does not shrink from describing the detailed calculations from which these lists are derived (the articles contain extensive – though not complete – sections of proofs) the surveys are far from being dry lists. He shows how a problem on estimating oscillatory integrals led him to start classifying functions; and by defining and computing an invariant (the 'Arnold index') applies the classification back to the original problem. In another paper, he relates the singularities of functions to those of projections of Lagrangian (and Legendre) submanifolds, and to the structure of caustics. The analysis of singularities of evolutes led to an extension of the work, in which singularities on the boundary of a manifold are investigated: an extension which in many ways completes the pattern set by the original.

Since catastrophe theory as such is not discussed in this volume, it is perhaps worth emphasizing here that Arnold's classification of simple singularities (A_k, D_k, E_k) contains and supersedes Thom's list of elementary catastrophes ($A_k: k \geqslant 5, D_4, D_5$). Moreover, the theory of Lagrangian maps coincides with the so called "catastrophe map". But Arnold goes further, and the relation of the theory of oscillating integrals to singularity theory, which he developed, has been aptly termed 'quantum catastrophe theory'.

The reader of this volume should not expect completeness: the results in these papers have stimulated much further work, and much yet remains to be discovered. But these surveys do contain Arnold's own analysis and synthesis of a decade's work on a fascinating topic, and it is with great pleasure that I introduce them to the reader.

SINGULARITIES OF SMOOTH MAPPINGS

V .I. ARNOL'D

The paper is based on a course of lectures on the local theory of singularities delivered in 1966 at a Summer School in Katsiveli and at the Moscow State University.

Contents

Chapter I

THE STRUCTURE OF SINGULARITIES

§1. Examples

The theory of singularities of smooth[2] mappings is concerned with local properties of differentiable mappings of differentiable manifolds,

1 The author is grateful to B. Malgrange, Yu.I. Manin, B. Morin, V.P. Palamodov and R. Thom for fruitful discussions and to S.M. Vishik, A.G. Kushnirenko and A.M. Leontovich for their assistance in preparing these lectures for the printer.

2 Here and in the sequel "smooth" and "differentiable" mean "infinitely differentiable". The tangent space at a point x in a manifold M is denoted by TM_x. The differential at x of a mapping $f:M\to N$ is denoted by f_x: $TM_x \to TN_{f(x)}$).

$$f\colon M^m \longrightarrow N^n,$$

invariant under diffeomorphisms

$$h\colon M \longrightarrow M, \qquad k\colon N \longrightarrow N.$$

EXAMPLE 1. Consider the mapping $f\colon \mathbf{R}^1 \to \mathbf{R}^1$ given by the formula $y = f(x) = x^2$ (fig. 1). In the neighbourhood of every $x \neq 0$, f is a diffeomorphism.

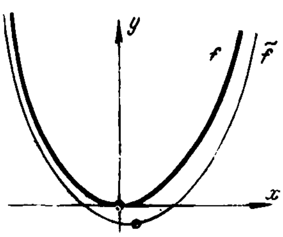

Fig. 1.

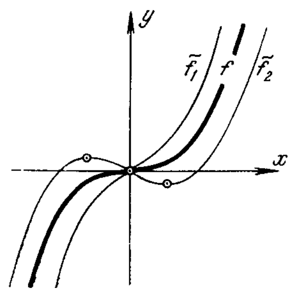

Fig. 2.

This is not the case in the neighbourhood of $x = 0$; f has a *singularity*, the differential of f degenerates at 0. This singularity is *stable*: every mapping $\tilde{f}$ near f whose derivatives are near to those of f has a similar singularity.

EXAMPLE 2. The mapping $y = x^3$ (fig. 2) also has a singularity, but it is *unstable*: for small deformations the singularity can vanish ($\tilde{f}_1$) or split into two ($\tilde{f}_2$).

We give now a general

DEFINITION 1. A differentiable mapping $f\colon M \to N$ is said to be *stable* if for any differentiable mapping $\tilde{f}\colon M \to N$ sufficiently close[1] to f there are diffeomorphisms $h\colon M \to M$, $k\colon N \to N$ close to the identity "converting" f to $\tilde{f}$, that is, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M & \xrightarrow{\tilde{f}} & N \end{array}$$

commutes. If there are *homeomorphisms* h and k that are near to 1_M and 1_N and make the diagram commute, then the mapping is said to be topologically stable.

[1] The topology on the space of differentiable mappings is that defined by these neighbourhoods of zero in the space of differentiable functions of local coordinates:

$$U(k, \varepsilon) = \left\{ \varphi(x)\colon \max_{|\alpha| \leq k} \left| \frac{\partial^{|\alpha|}\varphi}{\partial x^{\alpha}} \right| < \varepsilon \right\}$$

Slightly more complicated is the definition of stability at a point.

DEFINITION 2. A smooth mapping $f\colon U \to N^n$ defined on a neighbourhood U of a point x_0 in M^m is said to be *stable at* x_0 if for every mapping $\tilde{f}\colon U \to N^n$ sufficiently near to f there exist neighbourhoods $V \subset U \subset M^m$ of x_0 and $W \subset N^n$ of $y_0 = f(x_0)$, and diffeomorphic embeddings $h\colon V \to U$, $k\colon W \to N^n$ close to the identity and such that the diagram

$$\begin{array}{ccccccc} M^m \supset U \supset & V & \xrightarrow{f} & W & \subset N^n \\ & h\downarrow & & \downarrow k & \\ M^m \supset & U & \xrightarrow{\tilde{f}} & N^n & \end{array}$$

commutes. If, in addition, the h and k can be taken as homeomorphisms, we get the definition of *topological stability* at x_0; if f, $\tilde{f}$, h and k can be taken as real (complex) analytic functions, we get the definition of real (complex) analytic stability at x_0.

EXAMPLE 3. The implicit function theorem asserts that a mapping whose rank at x_0 is maximal is stable at x_0.

EXAMPLE 4. "Morse's Lemma" (see [1], p. 14) states that a mapping $\mathbf{R}^n \to \mathbf{R}^1$ given by a non-degenerate quadratic form

$$f(x) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2,$$

is stable at 0.

The ideal to which the theory of singularities strives is achieved in the special case of mappings $\mathbf{R}^n \to \mathbf{R}^1$ (Morse theory). The results of this theory that interest us can be stated as follows:

THEOREM 1. 1) *The stable mappings $f\colon M^m \to \mathbf{R}^1$ of a compact manifold M^m to the line form an everywhere dense set in the space of all smooth mappings.*

2) *A mapping f is stable if and only if the following two conditions are satisfied:*

M_1. *f is stable at every point (that is, every critical point of the function f is non-degenerate).*

M_2. *All critical values of f are distinct.*

3) *The mapping $f\colon M^m \to \mathbf{R}^1$ is stable at x_0 if and only if coordinates $x_1, x_2, \ldots, x_m, y$ can be introduced in neighbourhoods of x_0 in M^m and $y_0 = f(x_0)$ in $\mathbf{R}^1$ in such a way that f can be written in one of the $m + 2$ forms:*

M I. $y = x_1$,
M II$_k$. $y = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_m^2$ $(k = 0, 1, \ldots, m)$.

The following important case – the theory of two-dimensional manifolds – has received exhaustive study (Whitney [2]).

THEOREM 2. *The mapping $f\colon M^2 \to N^2$ is stable at the point x_0 if and only if it is equivalent in some neighbourhood of x_0 to one of three mappings* (fig. 3):

WI. $y_1 = x_1$, $y_2 = x_2$ *(regular point)*,
WII. $y_1 = x_1$, $y_2 = x_2^2$ *(fold)*,
WIII. $y_1 = x_1$, $y_2 = x_1x_2 - \frac{1}{3}x_2^3$ *(cusp) of a neighbourhood of 0 in the*

(x_1, x_2)-plane into a neighbourhood of 0 in the (y_1, y_2)-plane.

The stable mappings f: $M^2 \to \mathbf{R}^2$ of a compact surface into the plane form an everywhere dense set in the space of all smooth mappings.

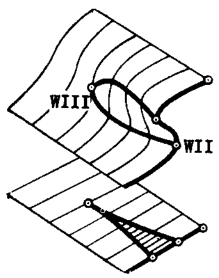

Fig. 3.

The smooth mapping f: $M^2 \to \mathbf{R}^2$ is stable if and only if the following two conditions are satisfied:

WI. *The mapping is stable at every point in M^2.*

WII. *The images of folds intersect only pairwise and at non-zero angles, whereas images of folds and cusps do not intersect.*

EXAMPLE 5. The Whitney mapping W. III. is stable (see fig. 4). Let us examine the structure more closely. Each line $x_1 = C$ goes into the line $y_1 = C$. But if $C < 0$, the image is monotone, while if $C > 0$ it is like $\tilde{f}_2$ in Example 2.

The differential f_* has the matrix

$$\begin{Vmatrix} 1 & 0 \\ x_2 & x_1 - x_2^2 \end{Vmatrix}.$$

The rank of f_* is 2 everywhere except on the parabolic fold $x_1 = x_2^2$. On the fold f_* has a kernel parallel to the x_2-axis. The image of the fold is the semi-cubic parabola $y_1 = x_2^2$, $y_2 = \frac{2}{3}x_2^3$. Every point inside the angle has three pre-images,

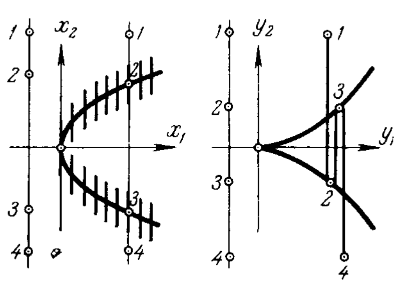

Fig. 4.

each point outside it has one. Note that the fold in the x-plane has no singularity, whereas its image in the y-plane has a singularity at 0. This is explained by the fact that the restriction of f to the parabola $x_1 = x_2^2$ has rank 0 at 0 (the kernel of f_* is tangent to the parabola at 0).

EXAMPLE 6. Consider the mapping $z \to z^2$ of the complex plane onto itself. The singularity at 0 is clearly not one of the types I, II, III. By Whitney's theorem, the mapping is unstable and small deformations can make it into a stable mapping having no singularities apart from folds and cusps. It turns out that it is enough to consider the nearby mappings $z \to z^2 + 2\varepsilon\bar{z}$.

The branch-point splits into three cusps, the circumference into a fold (Fig. 5). The image of the fold is a hypocycloid with three angles;

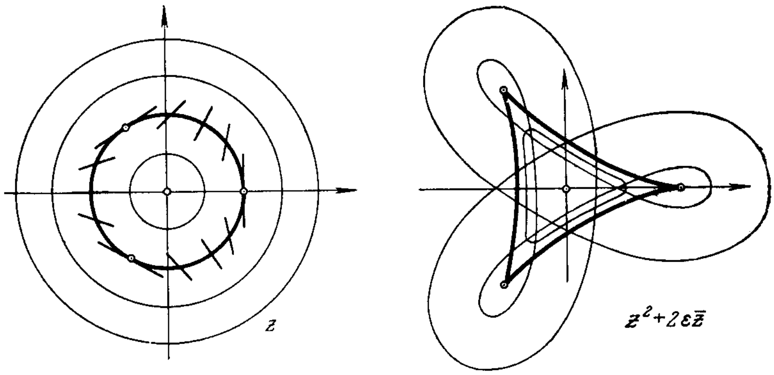

Fig. 5.

the points of the inside have four pre-images, those outside have two. The images of circles of large radius make two turns around the origin.

These examples might give rise to the hope that in higher dimensions every singularity is almost stable, and that stable singularities are easily classifiable. Indeed such a classification exists in dimensions less than six (Whitney [26]). However, the position is completely different in higher dimensions.

THEOREM 3 (Thom [3]). *For $n \geqslant 9$ there exist mappings $\mathbf{R}^n \to \mathbf{R}^n$ unstable at 0 but such that every sufficiently close mapping is unstable.*

Thus, for $n \geqslant 9$ the stable mappings do not form a dense set in the space of all mappings $M^n \to N^n$; there exist *continuous invariants* of smooth mappings f with respect to diffeomorphisms h and k. We give the proof of this theorem in §3.

Naturally the topological classification is coarser.

THEOREM 4 (Thom [4]). *There exists a topological classification of germs of smooth mappings f: $\mathbf{R}^m \to \mathbf{R}^n$ if a set of infinite codimension in the space of all germs is neglected.*[1]

More accurately this means the following. Denote by S the space of all germs of smooth mappings f: $\mathbf{R}^m \to \mathbf{R}^n$ at 0. Then

$$S = \bigcup_{\gamma \in \Gamma} S_\gamma + S_\infty,$$

where Γ is a countable set, each S_γ has finite codimension in S and

[1] The proof has not yet been published.

consists of topologically equivalent[1] germs; and S_∞ has infinite codimension.

We give here the definitions of some of the terms arising. The *germ* of the mapping f: $M^m \to N^n$ at x_0 is the equivalence class of mappings of neighbourhoods of x_0, φ: $U \to N^n$, $x_0 \in U \subset M^m$, where we say that two mappings φ_1: $U_1 \to N^n$, φ_2: $U_2 \to N^n$ are *equivalent* if they coincide on some neighbourhood of x_0 contained in $U_1 \cap U_2$. The *codimension* of a submanifold $A^k \subset B^n$ is $n-k$; that is, "the number of equations giving A locally". This definition carries over to the infinite-dimensional manifold S of germs when one uses the finite-dimensional approximations of S by the spaces of jets.

Two mappings φ_1: $U_1 \to N^n$, φ_2: $U_2 \to N^n$ are said to have *tangency of order* k at a point in $U_1 \cap U_2$ if $|\varphi_1(x) - \varphi_2(x)| = o(|x|^k)$ in some (and therefore in all) local coordinates. The *jet of order* k of the germ of a mapping f: $M^m \to N^n$ at x in M^m is the set of germs of mappings at x having tangency of order k with f at x. We denote this jet by $j_x^k(f)$. Clearly, $j_x^0(f) = f(x)$, $j_x^1(f)$ is defined by the differential f_x: $TM_x \to TN_{f(x)}$ of f at x.

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be choices of local coordinates in M and N, respectively. Then the jet $j_0^k(f)$ is defined by a segment of the Taylor series for f,

$$j_0^k(f) \sim f|_0 + \frac{\partial f}{\partial x}\bigg|_0 x + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\bigg|_0 x, x\right) + \ldots + \frac{1}{k!}\left(\frac{\partial^k f}{\partial x^k}\bigg|_0 x, \ldots, x\right).$$

Let $J_x^k(M^m, N^n)$ stand for the space of jets of order k of germs f: $M^m \to N^n$ at x. The preceding formula introduces into this space the structure of a finite-dimensional manifold of dimension

$$\dim J_x^k(M^m, N^n) = n + mn + \frac{m(m+1)}{2} n + \ldots + \binom{k+m-1}{m-1} n.$$

There is a natural mapping

$$\pi^k\colon S \to J_x^k,$$

associating with each germ at x its jet $j_x^k(f)$.

A set $S' \subset S$ is of *finite codimension* l, if for some k,

$$S' = (\pi^k)^{-1} J',$$

where J' is a submanifold of codimension l in J_x^k. In other words, *the set S' of germs has finite codimension l if it is given by l conditions on the Taylor coefficients of fixed order*. Further, S' has *infinite codimension if it lies in the intersection of a sequence of sets of increasing codimensions.*

The set of all jets of order k of germs of mappings f: $M^m \to N^n$ at different points forms a fibre bundle $J^k(M^m, N^n)$ with base $J^0(M, N) = M \times N$. It can be regarded as a vector bundle with the same base, and then we call it the *bundle of k-jets of mappings of M into N*. Finally, proceeding to the limit, as $k \to \infty$, in the sequence of projections $J^{k+1} \to J^k \to \ldots \to J^0$, we get the bundle of jets $J^\infty = J(M, N)$. The notation $J_x^\infty(M, N) = J_x(M, N)$ and $j_x^\infty(f) = j_x(f)$ has a similar meaning.

[1] Two germs f_1, f_2: $M^m \to N^n$ at x in M^m are topologically equivalent if there is a germ of a homeomorphism h: $M^m \to M^m$ fixing x and a germ of a homeomorphism k: $N^n \to N^n$ "taking $f_1(x)$ to $f_2(x)$", that is, such that the diagram

$$\begin{array}{ccc} M^m & \xrightarrow{f_1} & N^n \\ h\downarrow & & \downarrow k \\ M^m & \xrightarrow{f_2} & N^n \end{array}$$

commutes.

§2. The classes Σ^I

Let $f\colon M^m \to N^n$ be a smooth mapping and i an integer $\geqslant 0$.

DEFINITION 1. A point x of M^m lies in the set $\Sigma^i(f) \subset M$ if the kernel of the differential $f_*\colon TM^m_x \to TN^n_{f(x)}$ has dimension i. We say that f has singularity Σ^i at x or a singularity of class Σ^i.

EXAMPLE 1. For the Whitney singularity (fig. 6)

$$y_1 = x_1, \quad y_2 = x_1 x_2 - \frac{1}{3} x_2^3$$

we have

$$f_x = \begin{Vmatrix} 1 & 0 \\ x_2 & x_1 - x_2^2 \end{Vmatrix}, \quad \dim \operatorname{Ker} f_x = \begin{cases} 0 & \text{if} \quad x_1 \neq x_2^2, \\ 1 & \text{if} \quad x_1 = x_2^2. \end{cases}$$

Thus, Σ^0 is the whole plane excluding the parabola $x_1 = x_2^2$, Σ^1 is the parabola, the kernel is parallel to the x_2-axis; and for $i \geqslant 2$, Σ^i is empty.

We remark that the parabola Σ^1 is a smooth manifold which includes the cusp point 0. This point is distinguished by the fact that the kernel of f_0 is tangent to Σ^1 at it. In other words, *f restricted to* Σ^1, has rank 1 at all points other than 0. Thus, 0 lies in $\Sigma^1(f|\Sigma^1(f))$, and all other points on the parabola lie in $\Sigma^0(f\Sigma^1(f))$. We write $\Sigma^{i_2}(f|\Sigma^{i_1}(f))$ as $\Sigma^{i_1 i_2}(f)$. With this notation the Whitney cusp 0 lies in $\Sigma^{11}(f) \subset \Sigma^1(f)$.

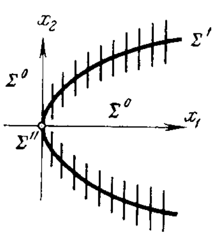

Fig. 6.

For any set $I = i_1, i_2, \ldots, i_n$ of integers the set $\Sigma^I(f)$ is defined by induction as follows.

DEFINITION 2. Let $\Sigma^I(f) = \Sigma^{i_1, \ldots, i_k}(f) \subset M$, M a smooth manifold. Then

$$\Sigma^{i_1, i_2, \ldots, i_k, i_{k+1}}(f) = \Sigma^{i_{k+1}}(f \mid \Sigma^I(f))$$

is the set of all points where the kernel of the differential of the restriction of f to $\Sigma^I(f)$ has dimension i_{k+1}.

REMARK. By the definition, the manifolds

$$M \supset \Sigma^{i_1} \supset \Sigma^{i_1 i_2} \supset \Sigma^{i_1 i_2 i_3} \supset \ldots$$

are embedded in one another. Thus, the kernels of the restrictions of f to these embedded submanifolds are also embedded in one another. So the *sequence of numbers* $i_1, i_2, \ldots$ comprising I must be non-decreasing, $m \geqslant i_1 \geqslant i_2 \geqslant i_3 \geqslant \ldots \geqslant 0$. If one of these inequalities is violated, then Σ^I is empty.

The set $\Sigma^I(f)$ is not necessarily a manifold, therefore the definition given above (due to Thom) does not give a definition of $\Sigma^I(f)$ for all f.

Boardman [5] has proposed a definition of $\Sigma^I(f)$ in terms of the space of jets. For any set $I = i_1, \ldots i_k$ of integers he defines a subset Σ^I of the space of k-jets $J^k(M^m, N^n)$, not depending on any mapping f (see below, p.11). He has proved:

THEOREM 1. *For any* $I = i_1, \ldots, i_k$ *the set* Σ^I *is a (not necessarily closed) submanifold of codimension* $\nu_I(m, n)$ *in* $J^k(M^m, N^n)$. *(The formula for* ν_I *is given below,* p.9).

The significance of Σ^I is that a "good" mapping f has singularity $\Sigma^I(f)$ at x in the sense of the preceding definition if and only if the jet of f at x lies in Σ^I.

DEFINITION 3. Let $f: M^m \to N^n$ be a smooth mapping. The *induced mapping* $\bar{f}: M^m \to J^k(M^m, N^n)$ associates with each point x of M^m the jet of f at x:

$$\bar{f}(x) = j^k_x(f).$$

A mapping is called "good" if its induced mapping $\bar{f}$ is transversal[1] on Σ^I.

Boardman has proved

THEOREM 2. 1) *If* f *is good, then* $\Sigma^I(f) = \bar{f}^{-1}(\Sigma^I)$; *that is,* $\Sigma^I(f)$ *is a manifold of codimension* $\nu_I(m, n)$ *in* M^m, *and* $x \in \Sigma^I(f)$ *if and only if the jet of* f *at* x *lies in* Σ^I.

2) *Every smooth mapping can be approximated, together with an arbitrary number of its derivatives, as closely as desired by a good mapping.*

Assertion 2) follows from 1) and Thom's transversality lemma. For $k = 1$ these results were established by Thom [3] and for $k = 2$ by Levine [3].

Let us consider the case $k = 1$ in more detail. A 1-jet of a smooth mapping taking x_0 in M^m to y_0' in N^n is given in local coordinates $x_1, \ldots, x_m$ in M^m and $y_1, \ldots, y_n$ in N^n by the matrix of the differential

$$f_{x_0} = \left\| \frac{\partial y_i}{\partial x_j}\Big|_{x_0} \right\| \qquad (i = 1, \ldots, n;\ j = 1, \ldots, m).$$

The $m \times n$ matrices form an mn-dimensional linear space L. Consider the set L_r of matrices of rank r. The numbers $k = m - r$ and $l = n - r$ can be called the coranks.

LEMMA 1. *The matrices of rank* r *form a smooth (non-closed) submanifold* L_r *in the space* L *of all* $m \times n$ *matrices, and the codimension of* L_r *is the product of the coranks:*

$$\dim L_r = mn - kl = mn - (m - r)(n - r).$$

PROOF. Since $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$ acts transitively on L_r, it is sufficient to consider the neighbourhood of the following matrix in L_r:

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1 Let A, B, C be smooth manifolds, $f: A \to B$ and $g: C \to B$ smooth mappings. Then f and g are said to be transversal if for every pair $a \in A$, $c \in C$ of points for which $f(a) = g(c) = b$ we have $f_*(TA_a) + f_*(TC_c) = TB_b$. If g is an *embedding*, then we speak of the transversality of f on the manifold C. Here $f^{-1}(C)$ is a submanifold of A and its codimension in A is the codimension of C in B (implicit function theorem).

Thom's "Transversality Lemma" asserts that *the set of mappings* $f: A \to B$ *transversal to a given mapping* $g: C \to B$ *is everywhere dense in the space of all differentiable mappings.* Further, *the set of mappings* f *such that* f *is transversal on an arbitrary submanifold of the space of jets is everywhere dense (and clearly open).*