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Excerpt
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Manifolds and differentiable structures

A manifold is a topological space which 'locally resembles' $\mathbb{R}^n$, the Euclidean space of real n-tuples $x = (x_1, \ldots, x_n)$ with the usual topology. Such spaces result in general, as we shall later see, as solution spaces of non-linear systems of equations, and many of the concepts of general topology have developed out of the study of these special spaces. The precise explanation is as follows:

(1.1) Definition. An *n-dimensional topological manifold* M^n is a Hausdorff topological space with a countable basis for the topology, which is locally homeomorphic to $\mathbb{R}^n$. The last condition means that, for each point $p \in M$, there exists an open neighbourhood U of p and a homeomorphism

$$h: U \to U'$$

onto an open set $U' \subset \mathbb{R}^n$ (Fig. 1).

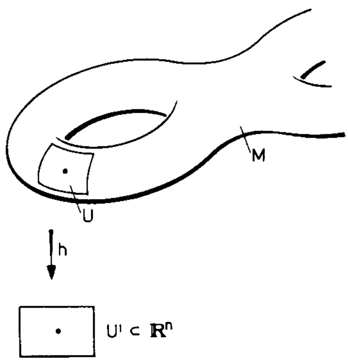

Fig. 1

The requirement that the space must be Hausdorff does not follow from the local condition as one might believe. As a counterexample one takes the real line $\mathbb{R}$, together with an additional point p, see Fig. 2, and defines the topology on $M = \mathbb{R} \cup \{p\}$ by saying that $\mathbb{R}$ is open and that the neighbourhoods of p are the sets $(U - \{0\}) \cup \{p\}$, where U is a neighbourhood of $0 \in \mathbb{R}$. Examples of topological manifolds (see Fig. 3) are:

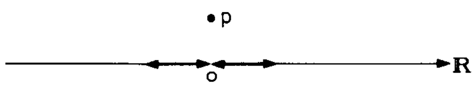

Fig. 2

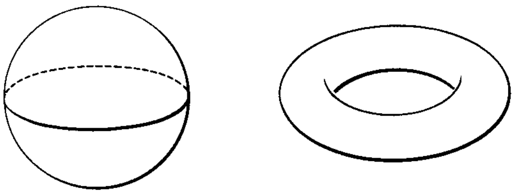

Fig. 3

every open subset of a Euclidean space;

the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$;

the torus or surface of an inner tube (1.8).

(1.2) Definition. If M^n is a topological manifold and $h\colon U \to U'$ a homeomorphism of an open subset $U \subset M$ onto the open subset $U' \subset \mathbb{R}^n$, then h is a *chart* of M and U is the associated *chart domain.* A collection of charts $\{h_\alpha \mid \alpha \in A\}$ with domains U_α is called an *atlas* for M if $\cup_{\alpha\in A} U_\alpha = M$.

Given two charts, both homeomorphisms h_α, h_β are defined on the intersection of their domains $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and one thereby obtains the chart transformation $h_{\alpha\beta}$ as a homeomorphism between open subsets of $\mathbb{R}^n$ by means of the commutative diagram:

$$\begin{array}{ccc} & U_{\alpha\beta} & \\ {\scriptstyle h_\alpha}\swarrow & & \searrow{\scriptstyle h_\beta} \\ U'_\alpha \supset h_\alpha(U_{\alpha\beta}) & \xrightarrow[h_{\alpha\beta}]{} & h_\beta(U_{\alpha\beta}) \subset U'_\beta \end{array}$$

in which $h_{\alpha\beta}$ is defined as $h_\beta \circ h_\alpha^{-1}$, see Fig. 4.

Occasionally, we shall find it useful to include the domain of definition of a map, particularly of a chart, in the notation, and thus we shall write (h, U) for a map $h\colon U \to U'$. If one were to consider the whole manifold as being formed by a glueing process from the chart domains, which one knows as well as one knows the open subsets of Euclidean space, then it is precisely the chart transformations which show how different chart domains are to be glued together. If, apart from the topological, one wishes to extend additional properties from open subsets of Euclidean space to manifolds by means of a

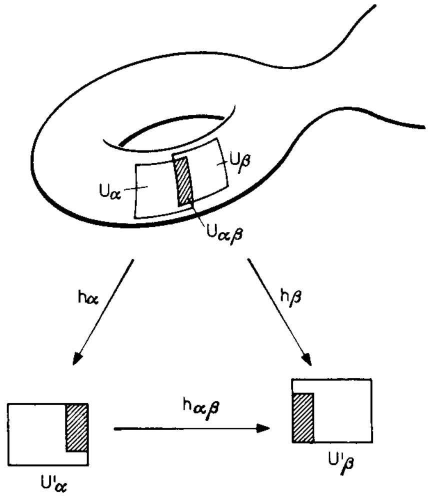

Fig. 4

suitable atlas, one must ensure that the definitions are independent of the particular choice of charts in the atlas, or that the property under consideration is independent of the chart transformations.

(1.3) Definition. An atlas of a manifold is called *differentiable*, if all its chart transformations are differentiable.

We shall always consider a differentiable mapping between open subsets of $\mathbb{R}^n$ to be a C^∞-mapping, that is, a mapping whose various (higher) partial derivatives exist and are continuous. Because, for the chart transformations $h_{\alpha\beta}$ (wherever the respective maps are defined), it is clear that

$$h_{\alpha\alpha} = \mathrm{Id}, \quad h_{\beta\gamma} \circ h_{\alpha\beta} = h_{\alpha\gamma},$$

it follows that

$$h_{\alpha\beta}^{-1} = h_{\beta\alpha}.$$

Therefore, the inverses of the chart transformations are also differentiable, and the chart transformations are diffeomorphisms.

If $\mathfrak{A}$ is a differentiable atlas on the manifold M, then the atlas $\mathfrak{D} = \mathfrak{D}(\mathfrak{A})$ contains precisely those charts for which every chart transformation with a chart from $\mathfrak{A}$ is differentiable. The atlas $\mathfrak{D}$ is then differentiable as well, since one can locally write a chart transformation $h_{\beta\gamma}$ in $\mathfrak{D}$ as a composition $h_{\beta\gamma} = h_{\alpha\gamma} \circ h_{\beta\alpha}$ of chart transformations for a chart $h_\alpha \in \mathfrak{A}$, and differentiability is a local property. As an element in the family of differentiable atlases, the atlas $\mathfrak{D}$ can obviously not be enlarged by the addition of further charts, and it is the largest differentiable atlas which contains $\mathfrak{A}$. Thus each differentiable atlas unequivocally determines a maximal differentiable atlas $\mathfrak{D}(\mathfrak{A})$, so that $\mathfrak{A} \subset \mathfrak{D}(\mathfrak{A})$; and $\mathfrak{D}(\mathfrak{A}) = \mathfrak{D}(\mathfrak{B})$ if and only if the atlas $\mathfrak{A} \cup \mathfrak{B}$ is differentiable. We formulate:

(1.4) Definition. A *differentiable structure* on a topological manifold is a maximal differentiable atlas. A *differentiable manifold* is a topological manifold, together with a differentiable structure.

In order to specify a differentiable structure on a manifold, one must specify a differentiable atlas and, in general, one will clearly not choose the maximal one, but preferably one as small as possible.

Henceforth we shall implicitly assume that all charts and atlases of a differentiable manifold with a differentiable structure $\mathfrak{D}$ are contained in $\mathfrak{D}$. In the notation, as usual, we employ the abbreviated form M, and not $(M, \mathfrak{D})$ for a differentiable manifold.

(1.5) Examples. (a) If $U \subset \mathbb{R}^n$ is an open subset, then the atlas $\{\mathrm{Id}_U\}$, which only contains the single chart $\mathrm{Id}\colon U \to U$, defines the usual differentiable structure. Furthermore, every homeomorphism $h\colon U \to U$ defines a differentiable atlas $\{h\}$, which gives the same differentiable structure if and only if h is a diffeomorphism. On an open subset of $\mathbb{R}^n$, one can therefore easily describe various differentiable structures for $n > 0$. However, as we shall yet see, using such atlases with only one chart $h\colon U \to U$, one does not obtain substantially different differentiable manifolds.

(b) The sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| := \sqrt{(x_1^2 + \ldots + x_{n+1}^2)} = 1\}$ possesses a differentiable atlas whose differentiable structure we shall always consider as introducing the standard structure on S^n. The chart domains are the sets

$$U_{kj} = \{x \in S^n \mid (-1)^j x_k > 0\},$$

the charts are

$$h_{kj}\colon U_{kj} \to \mathring{D}^n \quad \text{(the open solid ball)}$$

$$x \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}), \text{ see Fig. 5,}$$

so that the chart h_{kj} forgets the kth coordinate. It is easy to verify that this atlas is differentiable, since the map $h_{kj}^{-1}\colon \mathring{D}^n \to S^n$ has the kth coordinate (missing in $\mathring{D}^n$) $(-1)^j(1 - \Sigma_{i \neq k} x_i^2)^{1/2}$, which is clearly a differentiable function on $\mathring{D}^n$ in the usual sense; and h_{kj} results by restricting a differentiable mapping $\mathbb{R}^{n+1} \to \mathbb{R}^n$.

(c) The real projective space $\mathbb{R}\mathbf{P}^n$ is the quotient space of the sphere S^n under the equivalence relation defined by $x \sim -x$. A point $p \in \mathbb{R}\mathbf{P}^n$ is described by

$$p = [x] = [x_0, \ldots, x_n] = [-x_0, \ldots, -x_n], \quad \sum_{i=0}^{n} x_i^2 = 1,$$

and the equivalence relation identifies precisely the subsets $U_{k,0}$ and $U_{k,1}$ of the sphere. Therefore, the subsets

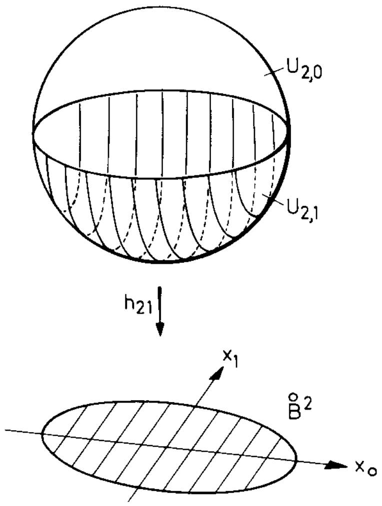

Fig. 5

$$U_k = \{[x] \in \mathbb{R}\mathbf{P}^n \mid x_k \neq 0\}$$

are open in $\mathbb{R}\mathbf{P}^n$, and one has charts

$$h_k\colon U_k \to \mathring{D}^n,\ [x_0, \ldots, x_n] \mapsto x_k \cdot |x_k|^{-1} \cdot (x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

for a differentiable atlas.

The projective spaces are examples of differentiable manifolds which arise naturally as abstract manifolds and not as subsets of Euclidean space. And, initially, it is not obvious that a projective space is homeomorphic to a subset of Euclidean space. One also obtains the topological manifold $\mathbb{R}\mathbf{P}^n$ when one identifies antipodal boundary points of the ball $D^n = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$, that is, forms the quotient for the equivalence relation '$x \sim -x$ for $|x| = 1$'. In this way, one can visualise the projective plane $\mathbb{R}\mathbf{P}^2$ as the result from glueing together a Möbius band B and a disc $A \cup C$ along their common boundary S^1, as in Fig. 6.

(d) An open subset of a differentiable manifold possesses a natural structure as a differentiable manifold.

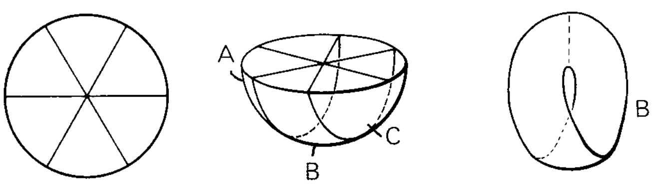

Fig. 6

Differentiable manifolds will be the subject of this book, more precisely, the category of differentiable manifolds. Its 'objects' are differentiable manifolds; its 'morphisms' are the differentiable mappings which we now define.

(1.6) Definition. A continuous mapping $f: M \to N$ between differentiable manifolds is termed *differentiable at the point* $p \in M$ if for some (and therefore for every) chart $h: U \to U'$, $p \in U$ and $k: V \to V'$, $f(p) \in V$ of M and N respectively, the composition $k \circ f \circ h^{-1}$ is differentiable at the point $h(p) \in U'$. Note that this mapping is defined in the neighbourhood $h(f^{-1}V \cap U)$ of $h(p)$, see Fig. 7. The mapping f is termed differentiable if it is differentiable at every point $p \in M$. In other words: one knows when one can call a mapping between chart domains of M and N differentiable, because these are identified by the charts with open subsets of Euclidean space, and locally a continuous mapping is always written as a mapping between chart domains. Independence from the particular choice of chart depends upon the fact that the chart transformations are differentiable.

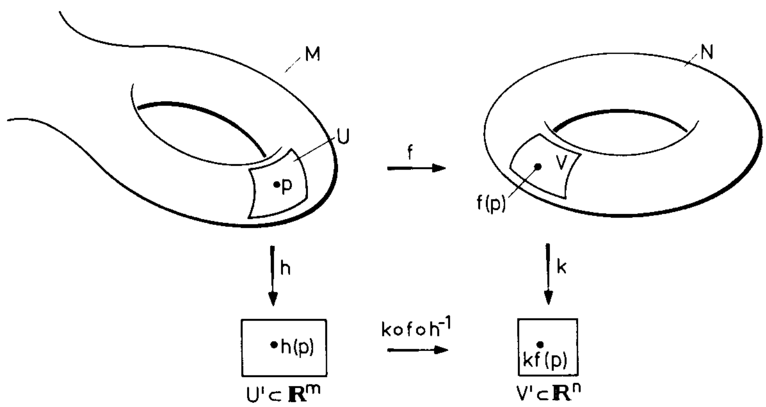

Fig. 7

Remark and notation. The identity mapping of a differentiable manifold is differentiable; the composition of differentiable mappings is differentiable. One assumes both these assertions in saying that differentiable manifolds and mappings form a category, the differentiable category which will be written C^∞ for short.

Correspondingly, let

$$C^\infty(M, N) := \text{the set of differentiable maps } M \to N;$$

$$C^\infty(M) := C^\infty(M, \mathbb{R}).$$

The composition of differentiable maps is therefore a map

$$C^\infty(M,N) \times C^\infty(L,M) \to C^\infty(L,N),\ (f,g) \mapsto f \circ g.$$

Many concepts arise in a category in a purely formal way; they are formulated using the maps of the category and their composition, as, for example, *isomorphism*, *sum*, and *product*.

(1.7) Definition. A *diffeomorphism* is an invertible differentiable map.

'Invertible', it is worth noting, means invertible in the differentiable category, therefore $f: M \to N$ is a diffeomorphism if there is a differentiable map $g: N \to M$, so that $f \circ g = \mathrm{Id}_N$ and $g \circ f = \mathrm{Id}_M$. This means, in other words: f is bijective and, also, f^{-1} is differentiable. We denote diffeomorphisms by '$\cong$'; they form the isomorphisms of the differentiable category.

A differentiable homeomorphism need not be a diffeomorphism, as is shown by the map $\mathbb{R} \to \mathbb{R}, x \mapsto x^3$.

For example, in (1.5(a)) we have introduced, in general, many distinct differentiable structures on an open subset $U \subset \mathbb{R}^n$, but the differentiable manifolds U with atlas $\{\mathrm{Id}\}$, and U with atlas $\{h\}$, are of course diffeomorphic; $h: U \to U$ is a diffeomorphism $(U, \{h\}) \to (U, \{\mathrm{Id}\})$ of the second onto the first. Thus, both manifolds are essentially the same in so far as their differential topology is concerned.

In contrast, the problem of constructing two distinct differentiable structures on a topological manifold, so that the resulting differentiable manifolds are not diffeomorphic, is very deep indeed. For example, the topological 7-sphere possesses exactly 15 mutually distinct non-diffeomorphic structures as a differentiable manifold. These are precisely 15 mutually distinct differentiable manifolds which are, however, all homeomorphic to the sphere S^7 (Kervaire & Milnor, 1963). Such results are far beyond the scope of this book.

Every chart $h: U \to U'$ of M is a diffeomorphism between U and U', where U' carries the standard structure as an open subset of $\mathbb{R}^n$ (1.5(d)), and the differentiable structure of M consists precisely of the set of all diffeomorphisms of open subsets of M with open subsets of $\mathbb{R}^n$.

The function $t \mapsto \tan((\pi/2)t)$ defines a diffeomorphism $(-1, 1) \to \mathbb{R}$.

Differential topology deals with those properties which remain constant under the action of diffeomorphisms. For local considerations, one can therefore always assume that one is dealing with an open subset of $\mathbb{R}^n$; instead of a function f on U, one considers $f \circ h^{-1}$ on U'; instead of an open subset $V \subset U$, the subset $h(V) \subset U'$; and so forth. Since images in $\mathbb{R}^n$ are given by their coordinates, one also often describes a chart of M around p in terms of a local coordinate system. The chart $h: U \to U'$ is written in components as $h = (h_1, \ldots, h_n)$, where the coordinate functions $h_i: U \to \mathbb{R}$ are differentiable functions; by translation in $\mathbb{R}^n$, one can further assume that $h(p) = 0 = (0, \ldots, 0)$ for a fixed point $p \in U$. Thus, in a neighbourhood U of p, after the introduction of a coordinate system, every point is uniquely determined by the values of the coordinate functions. Thus, for each point in U, one can assign coordinates

$$(x_1, \ldots, x_n)$$

with

$$(0, \ldots, 0) = \text{coordinates of } p.$$

A function on U is thus differentiable if and only if it is differentiable as a function of the coordinates in the usual meaning of the differential calculus.

In the differentiable category there are sums and products:

(1.8) Definition. The disjoint union of two n-dimensional differentiable manifolds M_1, M_2 is, in a natural way, a differentiable manifold expressed by $M_1 + M_2$, see Fig. 8. The topology is determined by the fact that both manifolds M_1, M_2 are open subsets of $M_1 + M_2$, and a differentiable atlas is the union of atlases of both manifolds.

The manifold $M_1 + M_2$ is called the *(differentiable) sum* of M_1 and M_2. One has canonical inclusions

$$i_\nu: M_\nu \to M_1 + M_2$$

as open subsets. A map $f: M_1 + M_2 \to N$ is then clearly differentiable if and only if both restrictions $f \circ i_\nu$ are differentiable; in other words one has a canonical bijection

$$C^\infty(M_1 + M_2, N) \to C^\infty(M_1, N) \times C^\infty(M_2, N), \ f \mapsto (f \circ i_1, f \circ i_2)$$

for every differentiable manifold N (*universal property of the sum*).

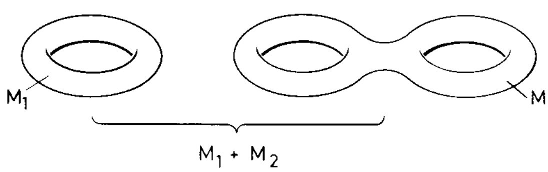

Fig. 8

Dually, one constructs the Cartesian product $M_1 \times M_2$ of two differentiable manifolds M_1, M_2 of dimensions n, k, and gives this the structure of a $(n + k)$-dimensional differentiable manifold which is called the *(differentiable) product* of M_1 and M_2. If $h_\nu: U_\nu \to U'_\nu$ are charts of the differentiable structure of M_ν, then

$$h_1 \times h_2: U_1 \times U_2 \to U'_1 \times U'_2 \subset \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$$

is a chart of $M_1 \times M_2$, and the set of all these charts defines the differentiable structure for $M_1 \times M_2$ (see Fig. 9). ($M_1 = M_2 = S^1$, with (p, q) a general point in the product.) One has canonical projections $p_\nu: M_1 \times M_2 \to M_\nu$

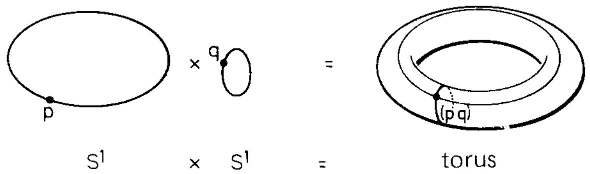

Fig. 9

and, analogously to the sum, a canonical bijection

$$C^\infty(N, M_1 \times M_2) \to C^\infty(N, M_1) \times C^\infty(N, M_2),\ f \mapsto (p_1 \circ f, p_2 \circ f)$$

for every differentiable manifold N (*universal property of the product*). The last remark states that a map into the product is differentiable if and only if both its components $f_\nu = p_\nu \circ f$ are differentiable; locally one maps into a chart domain $U_1 \times U_2$, and the composition with a chart

$$h_1 \times h_2 : U_1 \times U_2 \to U_1' \times U_2' \subset \mathbb{R}^{n+k}$$

is then differentiable if and only if both its components are differentiable.

Less canonical and, therefore, not so uniformly defined in the literature, is the concept of submanifold.

(1.9) Definition. A subset $N \subset M^{n+k}$ is called an *n-dimensional differentiable submanifold* of M if, for every point $p \in N$, there exists a chart around p

$$h : U \to U' \subset \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$$

so that

$$h(N \cap U) = U' \cap \mathbb{R}^n$$

where we consider $\mathbb{R}^n$ as $\mathbb{R}^n \times 0 \subset \mathbb{R}^n \times \mathbb{R}^k$.

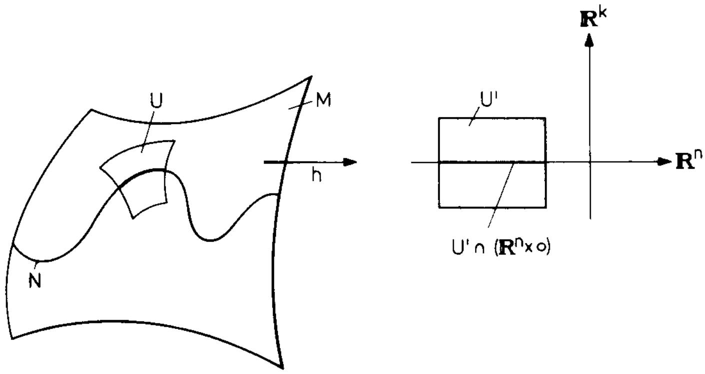

Fig. 10

The number $k = \dim M - \dim N$ is called the *codimension* of the submanifold. In short one says: locally the submanifold N lies in M as $\mathbb{R}^n$ lies in $\mathbb{R}^{n+k}$.

The definition is justified by the remark that there is a canonical differentiable structure on N. From a chart h, as in definition (1.9), one obtains a chart $h' = h | U \cap N \to U' \cap \mathbb{R}^n$, and the set of all these charts is a differentiable atlas for N, see Fig. 10.

(1.10) Definition. A differentiable map $f: N \to M$ is called an *embedding* if $f(N) \subset M$ is a differentiable submanifold, and $f: N \to f(N)$ is a diffeomorphism.

If N and M have the same dimension, then $f(N)$ is open in M, as definition (1.9) unmistakably shows, and the inclusion of an open subset is also an embedding. Otherwise, it is necessary that $\dim N < \dim M$. Every point $p \in M$ defines an embedding

$$i_p: N \to M \times N, \quad q \mapsto (p, q)$$

so that $p_2 \circ i_p = \mathrm{Id}_N$ and, similarly, every point $p \in M$ defines a projection $\pi_p: M + N \to M$, so that $\pi_p \circ i_1 = \mathrm{Id}_M$. The second factor, of course, behaves quite analogously; if $p \in M$ and $q \in N$, then $i_p(N)$ and $i_q(M)$ meet precisely in the point $(p, q) \in M \times N$, see Fig. 11.

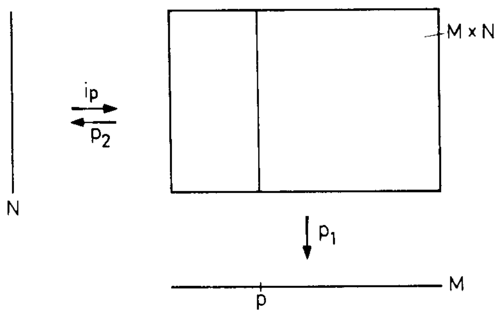

Fig. 11

(1.11) Exercises

1 Show that every (differentiable) manifold possesses a countable (differentiable) atlas.

2 Show that the sphere S^n possesses a differentiable atlas with precisely two charts. Also, one with only one chart?

3 Describe the chart transformation for the atlas of $\mathbb{R}\mathbf{P}^n$ in (1.5(c)), and show that it is differentiable.