CHAPTER 1. ABSTRACT SPECTRAL THEORY IN HILBERT SPACES.

In this chapter we give a short introduction into spectral theory of abstract unbounded operators of a Hilbert space. In sec. 1 we give a discussion of general facts on unbounded operators. In sec.2 we discuss the v.Neumann-Riesz theory of self-adjoint extension of hermitian operators. Sec.3 gives a general discussion of the abstract spectral theorem for unbounded self-adjoint operators. We discuss a proof of the spectral theorem in sec.4. Also, in sec.5 we discuss an extension of a result by Heinz and Loewner useful in the following. Finally an abstract result on Fredholm operators in a certain type of Frechet algebra related to a chain of Hilbert spaces generated by powers of a self-adjoint positive operator is discussed in sec.6. The typical 'HS-chain' is a chain of L^2-Sobolev spaces.

The chapter is self-contained and elementary, and only requires some familiarity with general concepts of analysis and functional analysis of bounded linear operators.

1. Unbounded linear operators on Banach and Hilbert spaces.

The term "(unbounded) linear operator" (between Banach spaces X and Y) is commonly used to denote any linear map $A:\mathrm{dom}\, A \to Y$ from a dense linear subspace dom A of X to Y. The space $\mathrm{dom}\, A \subset X$ then is called the domain of A. Here we distinguish between a linear map $X \to Y$,and a linear operator: A linear map $X \to Y$ by definition has its domain equal to X .

The term "unbounded linear operator" will be used with the meaning "not necessarily bounded linear operator", so that the bounded linear operators are special unbounded operators. A bounded linear operator, satisfying

$$(1.1) \qquad \sup \{\|Au\|/\|u\| : 0 \neq u \in \mathrm{dom}\, A\} < \infty ,$$

is necessasily continuous, hence admits a unique extension to X, in which dom A is dense. This is why we usually assume that a bounded linear operator also is a linear map.

The class of all such unbounded linear operators between two given spaces X and Y will be denoted by $P(X,Y)$. In particular the class $L(X,Y)$ of all continuous linear maps $X \to Y$ then is a subset of $P(X,Y)$.

Since unbounded linear operators are not linear maps from X to Y, (but only from their individual domain to Y) their sum and product needs the following special interpretation: For $A,B \in P(X,Y)$ and $C \in P(W,X)$ we define the sum $A+B \in P(X,Y)$ and the product $AC \in P(W,Y)$ by setting

(1.2) $\operatorname{dom}(A+B) = \operatorname{dom} A \cap \operatorname{dom} B$, $(A+B)u = Au+Bu$ for $u \in \operatorname{dom}(A+B)$,

and

(1.3) $\operatorname{dom} AC = \{u \in \operatorname{dom} C : Cu \in \operatorname{dom} A\}$, $(AC)u = A(Cu)$ for $u \in \operatorname{dom} AC$,

where it is assumed that dom(A+B) and dom AC are dense in X (or else, we will say that dom(A+B) or dom AC is not defined). Also we define $cA = (c \cdot 1)A$, with the identity operator $1 \in L(X,X)$.

A linear operator $A \in P(X,Y)$ is uniquely characterized by its graph, defined as the linear subspace

(1.4) $\operatorname{graph} A = \{(u;Au) : u \in \operatorname{dom} A\}$

of the cartesian product $X \times Y = \{(u;v) : u \in X, v \in Y\}$, where $(u;v)$ denotes the ordered pair. Vice versa, if for any linear subspace $T \subset X \times Y$ the set of all first components is dense in X, and if T does not contain elements of the form $(0;u)$ other than $(0;0)$, then a unique unbounded operator $A \in P(X,Y)$ is defined by setting

(1.5) $\operatorname{dom} A = \{u \in X : (u;v) \in T \text{ for some } v \in Y\}$, $Au=v$, for $u \in \operatorname{dom} A$,

and then we have $\operatorname{graph} A = T$.

For two linear operators $A,B \in P(X,Y)$ we shall say that A extends B (or that B is a restriction of A) if graph A $\supset$ graph B. We then will write $A \supset B$ (or $B \subset A$).

Notice that the cartesian product $X \times Y$ of two Banach spaces is a Banach space again, for example under the norm $\|(u;v)\| = \|u\| + \|v\|$. Therefore it is meaningful to speak of a closed subspace of $X \times Y$. An unbounded operator A is defined to be closed if its graph is a closed subspace of $X \times Y$. The class of all closed operators in $P(X, Y)$ is denoted by $Q(X, Y)$. It is clear that a continuous linear map $A \in L(X, Y)$ is closed, since $(u_k; Au_k) \to (u; v)$ implies $v = \lim Au_k = Au$, hence $(u;v) = (u;Au) \in$ graph A.

An operator $A \in P(X, Y)$ is called <u>preclosed</u> if the closure of graph A is a graph again. Then A^c with graph $A^c = (\text{graph } A)^{\text{closure}}$ is called the <u>closure</u> of A .

In the following we will be mainly interested in unbounded linear operators $A \in P(H) = P(H, H)$, where H is an infinite dimensional separable Hilbert space with inner product (u,v) and norm $\|u\| = \{(u,u)\}^{1/2}$. In that case the graph space $H \times H$ becomes a Hilbert space again, under the inner product and norm

$$(1.6) \quad ((u;w),(v;z)) = (u,v) + (w,z) \ , \ \|(u;w)\| = (\|u\|^2 + \|w\|^2)^{1/2} \ .$$

The following facts, regarding adjoint and closure all work for general Banach spaces X , Y , with proper amendments. However we will get restricted to the case $X = Y = H$, in all of the following. For an operator $A \in P(H)$ we will say that the (Hilbert space) <u>adjoint</u> $A^* \in P(H)$ exists if the space $T_A = J(\text{graph } A)^{\perp}$ is a graph again. Here $J: H \times H \to H \times H$ denotes the map $(u;w) \to (w;-u)$, and "$\perp$" denotes the orthogonal complement in the graph space (with respect to the inner product (1.6) . Then we define the adjoint $A^* \in P(H)$ of A by setting

$$(1.7) \qquad \text{graph } A^* = T_A \ .$$

Notice that A^*, if it exists, is closed, since all orthogonal complements are necessarily closed and since J is inverted by -J . It is clear that $A \subset B$ implies $B^* \subset A^*$, assuming that A^* and B^* exist. The proposition, below, translates the definition of the adjoint into a more transparent form. (The proof is left to the reader.)

<u>Proposition 1.1</u>. Assume that $A^* \in Q(H)$ exists, for some $A \in P(H)$. Then dom A^* consists precisely of all $u \in H$ for which there exists an element $v \in H$ such that

$$(1.8) \qquad (u,Aw) = (v,w) \ , \text{ for all } w \in \text{dom } A \ .$$

Moreover the element v thus defined for each $u \in \text{dom } A^*$ is uniquely determined, and we have $A^*u = v$.

<u>Proposition 1.2</u>. An operator $A \in P(H)$ admits an adjoint A^* if and

only if it is preclosed. Moreover, then A^* also admits an adjoint A^{**}, and the closure A^c of A equals A^{**}.

Proof. Let first $A \in P(H)$ have an adjoint $A^* \in Q(H)$. Then let $T = (\text{graph } A)^{\text{clos.}}$ contain the element $(0;z)$. It follows that there exists a sequence $w_k \in \text{dom } A$ with $(w_k;Aw_k) \to (0;z)$. Substitute $w = w_k$ in (1.8), and conclude that $(u,z) = 0$ for all $u \in \text{dom } A^*$, since the inner products in (1.8) allow a passing to the limit. Since dom A^* is dense, it follows that $z=0$, so that no element of the form $(0;z)$ is in T, except $(0;0)$. Also, $T \supset$ graph A, which implies that the set of first components of elements in T is dense. Therefore T indeed is a graph of some $A^c \in P(H)$, and A is preclosed.

Vice versa, let A be preclosed, and let again T be the closure of graph A. If the set V^* of all first components of elements in $T_A = J(\text{graph } A)^{\perp}$ (i.e., of all second components of $(\text{graph } A)^{\perp}$) is not dense in H then there exists $0 \neq z \in H$ with $((0;z),(u;v)) = (z,v) = 0$ for all $(u;v) \in (\text{graph } A)^{\perp}$. But this implies that $(0;z) \in (\text{graph } A)^{\perp\perp} = (\text{graph } A)^{\text{clos}} = T$. However, for preclosedness of A it is required that T does not contain such elements. Thus the set V^* must be dense in H. On the other hand if $(0;v) \in T_A = \text{graph } A^*$, then (1.8) yields $(v,w)=0$ for all $w \in \text{dom } A$, so that $v=0$, since dom A is dense. This shows that then indeed A^* is a well defined operator in $P(H)$, hence in $Q(H)$, q.e.d.

All continuous linear maps in $L(H)$ are closed, hence have an adjoint. Moreover $L(H)$ is adjoint invariant, and, for an $A \in L(H)$, the above adjoint coincides with the well known Hilbert space adjoint A^* of the bounded operator A.

Also, if $A \in P(H)$ is (pre-)closed, then $\gamma A+B$ is (pre-)closed, for every $B \in L(H)$, $0 \neq \gamma \in \mathbb{C}$. Then $(\gamma A+B)^c = \gamma A^c + B$, $(\gamma A+B)^* = \bar{\gamma} A^* + B^*$ in the sense of (1.2).

Our main interest, in the following, will focus on self-adjoint unbounded operators. Here an operator A is called self-adjoint if (A^* exists, and) $A^* = A$. First of all a self-adjoint operator allows a spectral decomposition, as a direct generalization of the principal axis transformation of a symmetric matrix. Second we will learn about important classes of differential operators which are unbounded self-adjoint operators, and therefore allow such a spectral decomposition. Third, we will show how results on unbounded non-selfadjoint differential operators can be achieved by " comparing" them with certain standard self-adjoint differential operators.

Note that a bounded operator (i.e., a continuous linear map) A is self-adjoint if and only if it satisfies the relation

$$(1.9) \qquad (u,Av) = (Au,v) \text{ for all } u,v \in \operatorname{dom} A ,$$

(where dom A = $\mathcal{H}$).A general unbounded operator A satisfying (1.9) needs not to be self-adjoint, because (1.9) just implies that $A^* \supset A$, not that $A^* = A$. Such an operator is called hermitian . If the closure of a hermitian operator A is self-adjoint then we speak of an essentially self-adjoint operator(i.e., $A^*=A^{**}$). Note that a hermitian operator A indeed has an adjoint:If $u_k \in \operatorname{dom} A$ $u_k \to 0$, $Au_k \to w$, then we may substitute $u=u_k$ into (1.9), for fixed v, and pass to the limit, resulting in $0 = (w,v)$, for all $v \in \operatorname{dom} A$. It follows that $w=0$, so that A is preclosed, and A^* exists, by prop. 1.2. Comparing (1.8) and (1.9), it then follows at once that $A \in P(\mathcal{H})$ is hermitian if and only if $A \subset A^*$.

If A is (essentially) self-adjoint, and B bounded hermitian then $\gamma A+B$ is (essentially) self-adjoint for all $0 \neq \gamma \in \mathbb{R}$.

More generally, two operators $A, B \in P(\mathcal{H})$ will be said to be in adjoint relation if

$$(1.10) \qquad (u,Av) = (Bu,v), \text{ for all } u \in \operatorname{dom} B , v \in \operatorname{dom} A .$$

The above conclusion, showing that hermitian operators have adjoints, can be repeated to prove that operators A, B in adjoint relation must have adjoints (both) ,and that,moreover, A and B are in adjoint relation if and only if $A \subset B^*$ (or if and only if $B \subset A^*$).

One of the first major problems occurring in our discussion of differential operators, in later sections, will be the construction of all self-adjoint extensions of a given hermitian operator. It turns out that not all hermitian operators possess self-adjoint extensions. On the other hand, the problem of characterizing all self-adjoint extensions was solved by v. Neumann [vN_1] and F. Riesz [Ri_1]. We will discuss the v. Neumann-Riesz theory in section 2, together with some other constructions of self-adjoint extensions.

2. Self-adjoint extensions of Hermitian operators.

In this section we discuss the v. Neumann-Riesz theory of self-adjoint extensions of hermitian operators.

It is at once clear that a hermitian operator A has a hermitian closure $A^c = A^{**}$, because $A \subset A^*$ implies $A^{**} \subset A^*$. Since a self-adjoint extension $B = B^*$ of A is necessarily closed, it also must be an extension of the closure A^{**}. Therefore, in looking for self-adjoint extensions of a given hermitian operator A we may look for such extensions of the closure, and thus assume that A is closed, without loss of generality.
Also if $A \subset B = B^*$, then $B^* = B \subset A^*$, so that every self-adjoint extension B of A is a restriction of A^*, as well: We have

$$(2.1) \qquad A \subset B = B^* \subset A^* .$$

Proposition 2.1. A hermitian operator A satisfies the identity

$$(2.2) \quad \|(A-\lambda)u\|^2 = \|(A-\mathrm{Re}\ \lambda)u\|^2 + (\mathrm{Im}\ \lambda)^2\|u\|^2 , \text{ for all } u\in \mathrm{dom}\ A ,\ \lambda\in\mathbb{C} .$$

Proof. We have $\|(A-\lambda)u\|^2 = ((A-\mu-i\nu)u, (A-\mu-i\nu)u) = ((A-\mu)u,(A-\mu)u) + \nu^2(u,u) - 2\mathrm{Re}\ ((A-\mu)u, i\nu u)$, where we have written $\mu = \mathrm{Re}\ \lambda$, $\nu = \mathrm{Im}\ \lambda$. Here the last term vanishes, due to $(Au,iu) + (iu,Au) = i((Au,u)-(u,Au)) = 0$, using that A is hermitian, q.e.d.

For a closed hermitian operator A it is implied by prop.2.1 that im $(A-\lambda)$ is closed for every nonreal $\lambda \in \mathbb{C}$. Indeed, consider a sequence $u_k \in \mathrm{dom}\ A$ such that $(A-\lambda)u_k \to v$. It follows that $(A-\lambda)(u_k-u_l) \to 0$, as $k, l \to \infty$. But (2.2) implies the inequality

$$(2.3) \qquad \|(A-\lambda)u\| \geq |\mathrm{Im}\ \lambda|\,\|u\| , \quad u \in \mathrm{dom}\ A .$$

Substituting $u=u_k-u_l$ into (2.3) yields $\|u_k-u_l\| \to 0$, since $\mathrm{Im}\ \lambda \neq 0$, by assumption. Hence $u_k \to u$ for some $u \in H$, and $(u_k;Au_k) \to (u;v)$ in $H\times H$. But graph A is closed ,since A is closed. Thus it follows that $(u;v) \in$ graph A, or, $u \in \mathrm{dom}\ A$, $v=Au$.

In the following we first consider the special case $\lambda = \pm i$. Since we have im $(A\pm i)$ closed, by the above, we obtain a pair of orthogonal direct decompositions

$$(2.4) \qquad H = \mathrm{im}(A\pm i) \oplus \mathcal{D}_\pm ,\quad \mathcal{D}_\pm = (\mathrm{im}(A\pm i))^\perp .$$

The two spaces $\mathcal{D}_\pm = \mathcal{D}_\pm(A)$ are called the defect spaces of the closed hermitian operator A , and their dimensions are called the defect indices of A . We write

(2.5) $\operatorname{def} A = (\dim(\operatorname{im}(A+i))^{\perp}, \dim(\operatorname{im}(A-i))^{\perp}) = (\nu_A^+, \nu_A^-)$.

Note that the defect spaces $\mathcal{D}_{\pm}$ are just the eigenspaces of the adjoint operator A^* to the eigenvalues $\pm i$: We have

(2.6) $$\mathcal{D}_{\pm} = (\operatorname{im}(A\pm i))^{\perp} = \ker(A^* \mp i) \quad .$$

Indeed, $f \in \mathcal{D}_+$,for example,amounts to $0 = (f,(A+i)u)$ for all $u \in \operatorname{dom} A$. We write this as $(f,Au) = (if,u)$, $u \in \operatorname{dom} A$, and compare with (1.8) , concluding that $f \in \operatorname{dom} A^*$, $A^*f = if$.

As another consequence of prop.2.1 we note that (2.2) ,for $\lambda = \pm i$, implies

(2.7) $$\|(A\pm i)u\| = (\|Au\|^2+\|u\|^2)^{1/2} = \|(u;Au)\| \ , \ u \in \operatorname{dom} A \ .$$

Note that graph A, as a closed subspace of $H\times H$ is a Hilbert space under the norm and inner product of $H\times H$. Moreover graph A is in linear 1-1-correspondence with dom A ,which may be used to transfer that Hilbert space structure of graph A to dom A . In other words, dom A is a Hilbert space under the (stronger) norm

(2.8) $$\|u\|_A = (\|u\|^2+\|Au\|^2)^{1/2} \ ,$$

with inner product

(2.9) $$(u,v)_A = (u,v) + (Au,Av) \ , \ u,v \in \operatorname{dom} A \ .$$

The latter is true for every closed operator $B \in \mathcal{Q}(H)$, not only for hermitian operators. In particular we may apply it to the adjoint $B=A^*$ of our closed hermitian operator A , obtaining a corresponding norm $\|u\|_{A^*}$ and inner product $(u,v)_{A^*}$, $u,v \in \operatorname{dom} A^*$. In fact, we then get $\|u\|_{A^*} = \|u\|_A$,for $u \in \operatorname{dom} A \subset \operatorname{dom} A^*$, and dom A appears as a closed subspace of dom A^* , under graph norm.

Note that (2.7) may be interpreted as follows: The two operators $(A\pm i)$ are isometries $\operatorname{dom} A \to \operatorname{im}(A\pm i) = \mathcal{D}_{\pm}^{\perp}$. In fact these isometries are 'onto'. Therefore $V = (A+i)(A-i)^{-1} : \operatorname{im}(A-i)\to\operatorname{im}(A+i)$ defines an isometry between the two closed subspaces of H .

Proposition 2.2. For a closed hermitian operator A we have

(2.10) $$\operatorname{dom} A^* = \operatorname{dom} A \oplus \mathcal{D}_+(A) \oplus \mathcal{D}_-(A) \quad ,$$

as an orthogonal direct decomposition of the Hilbert space dom A^* under its norm and inner product.

Proof. We already noticed that dom A is a closed subspace of dom A^*, under graph norm. The eigenspaces $\mathcal{D}_\pm$ of A^* are closed subspaces of $\mathcal{H}$, as nulspaces of the closed operators $A^* \mp i$. Moreover, on $\mathcal{D}_\pm$ we have $\|u\|_{A^*} = \sqrt{2}\|u\|$, so that $\mathcal{D}_\pm$ are also closed under graph norm. For $f_\pm \in \mathcal{D}_\pm$ one confirms that $(f_+,f_-)_{A^*} = 0$, using that $A^* f_\pm = \pm i f_\pm$. Also, for $u \in$ dom A, $(u,f_\pm)_{A^*} = (Au, \pm i f_\pm) + (u,f_\pm)$ $= \pm i((A \pm i)u, f_\pm) = 0$, so that the three spaces in the decomposition (2.10) are orthogonal. Suppose $f \in$ dom A^* satisfies $0 = (f,u)_{A^*}$ $= (A^*f, A^*u) + (f,u)$ for all $u \in$ dom A. Comparing this with (1.8) it is found that $A^*f \in$ dom A^*, hence $f \in$ dom $(A^*)^2$, and $(A^*)^2 f + f = 0$. One also may write this as $(A^*+i)(A^*-i)f = (A^*-i)(A^*+i)f = 0$. In particular, $f \in$ dom $(A^*+i)(A^*-i)$ = dom $(A^*-i)(A^*+i)$, in the sense of (1.3). Now we write $f = (A^*+i)f/2i - (A^*-i)f/2i = f_+ + f_-$, noting that $(A^* \mp i) f_\pm = 0$. This proves that every f orthogonal to dom A, under graph inner product of A^*, is in $\mathcal{D}_+ \oplus \mathcal{D}_-$, and thus completes the proof.

Corollary 2.3. A closed hermitian operator A is self-adjoint if and only if its defect indices vanish (i.e. def A = 0). Or, equivalently, if and only if im $(A \pm i) = \mathcal{H}$, for both, "+" and "-".

Indeed, $A = A^*$ implies dom A = dom A^*, so that (2.10) gives $\mathcal{D}_\pm = \{0\}$, hence def A = 0. Vice versa, if def A = 0, (2.10) gives dom A = dom A^*, hence $A^* = A$, since $A^* \supset A$, q.e.d.

We now state the v. Neumann-Riesz extension theorem.

Theorem 2.4. The closed hermitian extensions B of a given closed hermitian operator A are in 1-1-correspondence with the extensions $W : \mathcal{W}_- \to \mathcal{W}_+$ of the isometry $V = (A+i)(A-i)^{-1}$: im(A-i) → im(A+i) as an isometry between the closed subspaces $\mathcal{W}_\pm \supset$ im$(A \pm i)$. This correspondence is established by assigning to $B \supset A$ the operator $W = V_B = (B+i)(B-i)^{-1}$ (which is an isometry extending V between the spaces $\mathcal{W}_\pm$ = im$(B \pm i) \supset$ im$(A \pm i)$). Vice versa, given an isometry $W : \mathcal{W}_- \to \mathcal{W}_+$ extending the isometry V, one must observe that W, as an extension of V, is determined by its restriction $W_0 = W|\mathcal{W}^0_-$, where $\mathcal{W}^0_- = \mathcal{W}_- \cap (\text{im}(A-i))^\perp = \mathcal{W}_- \cap \mathcal{D}_-$ is a subspace of the defect space $\mathcal{D}_-$, and where W_0 is just any isometry $\mathcal{W}^0_- \to \mathcal{W}^0_+ = \mathcal{W}_+ \cap \mathcal{D}_+ \subset \mathcal{D}_+$. Then we have the closed hermitian extension B given by

(2.11) $\operatorname{dom} B = \operatorname{dom} A \oplus \{W_0\phi - \phi : \phi \in \mathcal{W}^0_-\}$, $B(W_0\phi-\phi)=i(W_0\phi+\phi)$,

where the direct sum again is orthogonal in $(.,.)_{A^*}$.

The proof is almost self-explanatory. It is clear from the above that $W = (B+i)(B-i)^{-1}$ is an isometric extension of V , for every closed hermitian extension B of A . Vice versa, that W_0 determines W , as described, follows from the well known fact that isometries preserve orthogonality. Then, of course, the operator B , if it exists,should satisfy

(2.12) $W(B-i)u = (B+i)u$, $u \in \operatorname{dom} B = \operatorname{dom} A \oplus \mathcal{Z}_0$,

with a certain subspace $\mathcal{Z}_0 \subset \mathcal{D}_+ \oplus \mathcal{D}_-$, because $\operatorname{dom} A \subset \operatorname{dom} B \subset \operatorname{dom} A^*$ and due to (2.10) . For $u \in \mathcal{Z}_0$ let $\phi = Bu-iu$, $\chi = Bu + iu = W_0\phi$. It follows that $u = (W_0\phi-\phi)/2i$, $Bu = (W_0\phi+\phi)/2$, in agreement with (2.11). Now one simply must verify that the operator B of (2.11) is closed and hermitian. The closedness follows if we show that $\mathcal{Z}_0 = \{W_0\phi-\phi : \phi \in \mathcal{W}^0_- \}$ is a closed space, under graph norm of A^* . But we have

(2.13) $\|W_0\phi-\phi\|_{A^*}^2 = \|W_0\phi-\phi\|^2 + \|W_0\phi+\phi\|^2 = 2\|W_0\phi\|^2 + 2\|\phi\|^2 = 4\|\phi\|^2$,

which shows that the map $\phi \to W_0\phi-\phi$, taking $\mathcal{W}^0_-$ onto $\mathcal{Z}_0$, is, in fact an isometry (up to the factor 4). Since $\mathcal{W}^0_-$ is closed, $\mathcal{Z}_0$ also is closed. To verify that B of (2.11) is hermitian is only a calculation; since we know that $B|\operatorname{dom} A = A$ is hermitian one must show that $(A^*u,v)=(u,A^*v)$ for all $u,v \in \mathcal{Z}_0$, and for $u \in \operatorname{dom} A$, $v \in \mathcal{Z}_0$. Both follow trivially, q.e.d.

Theorem 2.4 has the following important consequence.

Corollary 2.5. A closed hermitian operator A admits a self-adjoint extension if and only if $\operatorname{def} A = (\nu,\nu)$, with $\nu=0,1,2,\dots,\infty$ arbitrarily given. In other words, we must have

(2.14) $\operatorname{codim} \operatorname{im}(A+i) = \operatorname{codim} \operatorname{im}(A-i)$.

Then every self-adjoint extension B of A is obtained by picking an arbitrary isometry $W_0 : \mathcal{D}_- \to \mathcal{D}_+$,between the two defect spaces (2.6), and then defining B with (2.11) , and W_0 , with $W^0_\pm = \mathcal{D}_\pm$.

The proof is evident.

Although the v.Neumann-Riesz theory completely clarifies the problem of self-adjoint extensions, other criteria are useful, of course, because the construction of isometries between defect spaces is not always practical. In particular not every closed hermitian operator A satisfies the condition (2.14), so that a self-adjoint extension need not always to exist. There are two well known general criteria giving existence of self-adjoint extensions. Shortly, 'real' hermitian operators as well as 'semi-bounded' hermitian operators always have self-adjoint extensions.

The concept of real operator refers to a given involution $u \to \bar{u}$ of the Hilbert space $\mathcal{H}$. In most applications we will have $\mathcal{H} = L^2(X,d\mu)$ with some measure space X and measure $d\mu$, and then refer to the complex conjugation $u(x) \to \bar{u}(x)$ of the complex-valued function $u(x) \in \mathcal{H} = L^2$. However, one may think of an abstract space $\mathcal{H}$ and an involution map $u \to u^-$, with the properties

$$(2.15)\qquad (u^-)^- = u \ , \ (c_1u+c_2v)^- = \bar{c}_1u^-+\bar{c}_2v^- \ ,$$
$$(u^-,v^-) = (v,u) \ , \ \text{for } u,v \in \mathcal{H} \ , \ c_j \in \mathbb{C} \ , \ j=1,2.$$

Then a real operator is defined as an operator $A \in P(\mathcal{H})$ satisfying

$$(2.16)\qquad (\text{dom } A)^- = \text{dom } A \ ,\text{and} \quad (Au)^- = Au^- \ , \ \text{for all } u \in \text{dom } A \ .$$

Now, if a closed hermitian operator A is real with respect to any such involution of $\mathcal{H}$, then one confirms at once that

$$(2.17)\qquad \mathcal{D}_+^- = \{u^-: u \in \mathcal{D}_+\} = \mathcal{D}_- \ .$$

Indeed if $f \in \mathcal{D}_+$, i.e., $(f,(A+i)u)=0$ for all $u \in \text{dom } A$, then we get $0 = ((A+i)u,f)=(f^-,((A+i)u)^-)=(f^-,(A-i)u^-)$, hence $f^- \in (\text{im}(A-i))^\perp$, using (2.16). This conclusion may be reversed, so that (2.17) follows. Also it is clear that $\mathcal{D}_+$ and $\mathcal{D}_+^-$ have the same dimension.

We have proven:

Proposition 2.6. A closed hermitian operator A which is real with respect to some involution of $\mathcal{H}$ has equal defect indices and hence admits a self-adjoint extension.

A hermitian operator $A \in P(\mathcal{H})$ is called semi-bounded below, if there exists a real constant c such that