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INTRODUCTION TO PART B

An inexhaustible source of algebraic topology is the homotopy classification problem. If we have a space we would like to know a list of algebraic invariants which determine the homotopy type of the space. If we have a map we would like to characterize the map up to homotopy by algebraic invariants. Moreover, if the set of homotopy classes $[X, G]$ is a group, for example if G is a topological group, we would like to determine the group structure of the set as well. Helpful tools for these problems are functors from the topological category to an algebraic category like homology, cohomology, homotopy groups etc. However, the known functors give us only rather crude algebraic pictures and almost nothing is known about the image categories of these functors.

There are two opposite directions in which the problem can be pursued, namely rational homotopy theory and stable homotopy theory. Both are studied with great energy. Indeed, research camps seem to have formed - on one side of the front are those mathematicians who think a rational space is the most natural object, on the other side those for whom a spectrum is the most natural object to start with. At the time of J. H. C. Whitehead people were interested in finite polyhedra. Soon they realized that the calculation of homotopy groups of spheres is a deep and fundamental obstacle to solving the classification problems. Rationally these groups were computed by Serre. Modulo a prime there are partial results, but the nature of the groups is essentially still unknown. Moreover, we have a complete rational solution of the homotopy classification problem in the results of Quillen and Sullivan. They have shown that the homotopy categories of differential graded Lie algebras over $\mathbb{Q}$ and of differential graded commutative algebras over $\mathbb{Q}$ are equivalent to the homotopy category of 1-connected rational spaces. In view of this we may now well doubt whether stable homotopy theory is a 'first approximation to homotopy theory' as it was conceived to be by J. H. C. Whitehead

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and Spanier (in Proc. Nat. Acad. Sci. 39 (1953), 655-60). Instead, we might try to understand the homotopy classification problem by extending the rational solution of the problem to a solution over a subring R of the rationals. It is one purpose of this book to initiate such an investigation.

The only result in the literature which illustrates our approach is the Hilton-Milnor theorem on the graded homotopy group of a one-point union of spheres. Rationally this group is just a free graded Lie algebra. Hilton showed that over the integers we still have a direct sum decomposition of these groups in terms of a basis in a free Lie algebra and in terms of homotopy groups of spheres. We believe in fact that over the integers this group should also be a free object in a suitable category. It developed that a description of even this category requires a formidable apparatus. However, for a subring R of the rationals $\mathbb{Q}$ with 1/2, $1/3 \in R$ a category as in the following problem has an elegant characterization. Let Top_ε be the homotopy category of ε-connected spaces.

(1) Problem. Construct a category Lie_M such that the functor of homotopy groups

$$L(\,.\,, R): Top_1 \to Lie_M \quad \text{with}$$

$$L(X, R) = \pi_*(\Omega X) \otimes R \quad \text{(endowed with suitable algebraic structure)}$$

maps a one-point union of spheres V to a free object L(V, R) in Lie_M.

Clearly Whitehead or Samelson products give the graded module $L(X, R) = \pi_*(\Omega X) \otimes R$ the structure of a graded Lie algebra over R. Only for $R = \mathbb{Q}$ do we know that $L(V, \mathbb{Q})$ is also a free Lie algebra. For $R \neq \mathbb{Q}$ we have to introduce additional structure on L(X, R) to obtain the objects in the category Lie_M required by problem (1). At first sight this problem might appear to be a merely formal question. The solution nevertheless is essential to our notion of extending rational homotopy theory to a theory over a subring R of $\mathbb{Q}$. The category Lie_M should play a role in homotopy theory over R similar to that of the category $Lie_{\mathbb{Q}}$ of graded rational Lie algebras in rational homotopy theory. Clearly the non-rational theory is enormously more complicated than the rational one. Still, the study of the rational situation can serve as a guide. We

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will give various examples to illustrate this.

Rational homotopy theory can be developed in two ways dual to each other, namely via the cofibration or Lie algebra method of Quillen and by the fibration or commutative algebra method of Sullivan. It turned out that the cohomology functor $H^*(. , \mathbb{Q})$ has properties completely dual to those of the homotopy functor $L(. , \mathbb{Q})$. This leads to our next problem.

(2) Problem. Is there a category Div alg_M and a functor

$$M(. , R) : \text{Top}_o \to \text{Div alg}_M$$

with properties dual to those of the functor $L(. , R)$ so that this duality extends the known duality of the rational functors

$$M(. , \mathbb{Q}) = H^*(. , \mathbb{Q}) \text{ and}$$

$$L(. , \mathbb{Q}) = \pi_*(\Omega\, .) \otimes \mathbb{Q}\ ?$$

Clearly the formulation of this problem is not very precise. It expresses only our feeling of what it would be nice to have. We will show that there is such a functor $M(. , R)$, which we call the spherical cohomotopy functor. For a finite dimensional polyhedron X the graded R-module $M(X, R)$ is given by the set of homotopy classes

$$M^n(X, R) = [X, \Omega^n_R]$$

where for the R-local n-sphere S^n_R we set

$$\Omega^n_R = \begin{cases} S^n_R & n \text{ odd} \\ \Omega\Sigma S^n_R & n \text{ even} > 0\,. \end{cases}$$

By a result of Serre we know that $\Omega^n_{\mathbb{Q}} = K(\mathbb{Q}, n)$ is an Eilenberg-MacLane space. It is well known that for a product $P_{\mathbb{Q}}$ of such Eilenberg-MacLane spaces the cohomology algebra $H^*(P_{\mathbb{Q}}, \mathbb{Q})$ is a free graded commutative algebra. More generally, we found that for a product P_R of spaces Ω^n_R the algebra $M(P_R, R)$ is also a free object in the category Div alg_M, which we construct. This category is appropriate for homotopy theory

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over R and generalizes the category of rational commutative algebras.

Although the construction of the categories Lie_M and Div alg_M is quite intricate, it can be sketched as follows. The homotopy groups of spheres give us the double graded R-module $M = M_R$ with

$$M_R^{n,m} = M^n(S^m, R) = \begin{cases} \pi_m(S^n) \otimes R & n \text{ odd} \\ \pi_{m+1}(S^{n+1}) \otimes R & n \text{ even} \end{cases} .$$

This module with additional structure (namely smash product, higher-order Hopf invariants γ_p for each prime p and units $e^r \in M_R^{r,r} = R$) is an object in the category $Coef_R$ of coefficients. It turns out that there is an associative tensor product $\tilde{\otimes}$ in this category $Coef_R$ which we can use to define the notion of a monoid in $Coef_R$. We show that in fact the composition $\odot$ of maps gives M_R the monoid structure

$$\odot : M_R \tilde{\otimes} M_R \to M_R$$

in $Coef_R$. For the categories

$$\begin{cases} Lie_R & = \text{category of graded Lie algebras over } R \\ \text{Div } alg_R & = \text{category of graded commutative algebras over } R \\ & \quad \text{with divided powers} \end{cases}$$

we construct bifunctors

$$(3) \quad \begin{cases} Lie_R \times Coef_R \xrightarrow{\tilde{\otimes}} Lie_R \\ Coef_R \times \text{Div } alg_R \xrightarrow{\tilde{\otimes}} \text{Div } alg_R \end{cases}$$

which are associative with respect to the tensor product $\tilde{\otimes}$ in $Coef_R$. With these 'twisted' products $\tilde{\otimes}$ we define actions of the monoid M_R to be morphisms

$$(4) \quad \begin{cases} \odot : L \tilde{\otimes} M_R \to L & \text{in } Lie_R \\ \odot : M_R \tilde{\otimes} A \to A & \text{in Div } alg_R \end{cases}$$

which are associative with respect to the monoid structure $\odot$ on M_R.

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The objects of Lie_M and $Divalg_M$ are now just the objects of Lie_R and $Divalg_R$ together with such an action. The morphisms are just the equivariant maps in Lie_R or $Divalg_R$ respectively.

In this way we replace the coefficient module $\mathbb{Q} = M_{\mathbb{Q}}$ of rational homotopy theory by the coefficient module $M = M_R$ needed in homotopy theory over R. Here we are not deterred by not knowing explicitly the homotopy groups of spheres $M_R^{n,m}$. We just clarify the 'primary' algebraic structure of these groups, namely their structure as a monoid in $Coef_R$. The construction of the categories Lie_M and $Divalg_M$ depends only on this primary structure. We will prove that there exist free objects in these categories. This now allows us to formulate the first basic classification result.

(5) **Theorem.** (A) The full subcategory of Top_1 of spaces homotopy equivalent to a one-point union of finitely many R-local spheres S_R^n is, via the functor $L(. , R)$, equivalent to the full subcategory of Lie_M of finitely generated free objects.

(B) The full subcategory of Top_0 of spaces homotopy equivalent to a product of finitely many spaces Ω_R^n is, via the functor $M(. , R)$, equivalent to the full subcategory of $Divalg_M$ of finitely generated free objects.

For $R = \mathbb{Q}$ this is a well known result of rational homotopy theory, in fact for $R = \mathbb{Q}$ it is the restriction of the equivalences of Sullivan and Quillen to the case of zero differentials.

We investigate the connection of the homotopy functors $L(. , R)$ and $M(. , R)$ with the corresponding homology and cohomology functors. That is, we consider the Hurewicz and degree maps, which are natural transformations

$$(6) \qquad \begin{cases} \Phi = \Phi_Y & : L(Y, R) \to PH_*(\Omega Y, R) \\ \deg = \deg_X & : M(X, R) \to H^*(X, R) \end{cases}$$

of Lie algebras and algebras respectively. Here we restrict the spaces X and Y to those for which $H_*(\Omega Y, R)$ and $H^*(X, R)$ are free R-modules of finite type. Then the Lie algebra $PH_*(\Omega Y, R)$ of primitive elements is

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defined. For $R = \mathbb{Q}$ the Hurewicz map Φ is an isomorphism by the Milnor-Moore theorem. Dually, deg is also an isomorphism for $R = \mathbb{Q}$. For $R \neq \mathbb{Q}$ the behaviour of Φ and deg is unknown. Therefore we consider only spaces X and Y for which either Φ or deg respectively is still surjective, or else even admits a right inverse in the categories Lie_R or $Divalg_R$ respectively.

For the twisted products $\tilde{\otimes}$ in (3) we show

(7) Theorem. (A) If Φ_Y admits a right inverse we have an isomorphism in Lie_M

$$L(Y, R) \cong PH_*(\Omega Y, R) \tilde{\otimes} M_R$$

(B) If $\deg_X$ admits a right inverse we have an isomorphism in $Divalg_M$

$$M(X, R) \cong M_R \tilde{\otimes} H^*(X, R).$$

Clearly (A) corresponds to the Milnor-Moore theorem for $R = \mathbb{Q}$. For $R \neq \mathbb{Q}$ the Hilton-Milnor theorem, as well as the results of G. J. Porter on homotopy groups of a fat wedge of spheres, are further illustrations of (A).

Next we study the R-localization of the group $[\Sigma X, Y] = [X, \Omega Y]$, which we assume to be nilpotent. Our results are also applicable to the study of the group $[X, G]$ where G is a topological group. We define a bifunctor

$$\exp_M : Divalg_M \times Lie_M \to \text{Category of groups}$$

which is essentially the exponential group on a Lie algebra. Furthermore we obtain a natural homomorphism

$$\rho : \exp_M(M(X, R), L(Y, R)) \to [\Sigma X, Y]_R$$

and we prove

(8) Theorem. If Φ_Y or $\deg_X$ is surjective, the homomorphism ρ is an isomorphism.

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If Φ_Y or $\deg_X$ even admits a right inverse, we can replace the coefficients M by R. In fact, since we have isomorphisms

$$\exp_M(M \tilde{\otimes} A,\ L) = \exp_R(A,\ L)$$
$$\exp_M(B,\ K \tilde{\otimes} M) = \exp_R(B,\ K)$$

we obtain from (8) and (7)

(9) **Theorem.** (A) _If $\deg_X$ admits a right inverse, we have an isomorphism_

$$\exp_R(H^*(X,\ R),\ \ \pi_*(\Omega Y) \otimes R) \cong [\Sigma X,\ Y]_R\ .$$

(B) _If Φ_Y admits a right inverse, we have an isomorphism_

$$\exp_R(M(X,\ R),\ PH_*(\Omega Y,\ R)) \cong [\Sigma X,\ Y]_R\ .$$

(C) _If $\deg_X$ and Φ_Y both admit right inverses, the group $[\Sigma X,\ Y]_R$ depends only on the cohomology algebra $H^*(X,\ R)$ and on the homology Lie algebra $PH_*(\Omega Y,\ R)$._

Clearly for $R = \mathbb{Q}$ the propositions (A), (B) and (C) coincide. For $R = \mathbb{Q}$ theorem (9) is equivalent to

(10) **Theorem.** _There is a natural isomorphism of rational nilpotent groups_

$$[\Sigma X,\ Y]_{\mathbb{Q}} \cong \exp \operatorname{Hom}(H_*(X,\ \mathbb{Q}),\ \ \pi_*(\Omega Y) \otimes \mathbb{Q}).$$

Here the $\mathbb{Q}$-vector space of degree zero homomorphisms

$$\operatorname{Hom}(H_*(X,\ \mathbb{Q}),\ \ \pi_*(\Omega Y) \otimes \mathbb{Q})$$

has in a natural way the structure of a nilpotent rational Lie algebra and exp denotes the group structure on this rational vector space given by the Baker-Campbell-Hausdorff formula. With a certain amount of work, formula (10) can also be derived from the rational homotopy theories of Quillen or Sullivan. However, the formula does not appear in the literature. We give a different type of proof which is based only on the old

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result of Serre that the rational n-sphere $S^n_{\mathbb{Q}}$ is an Eilenberg-MacLane space if n is odd.

As a generalization of (10) we obtain, for example, from (9) (A) and (3.9) in chapter V

(11) Theorem. Let $H_*(X, R)$ be a finitely generated free R-module. Let X be connected and let G be a connected topological group. Then there is a natural isomorphism

$$[X, G]_R = \exp_R(H^*(X, R), \pi_*(G) \otimes R)$$

of R-local nilpotent groups if we assume that R contains 1/p for all primes p with

$$p < \tfrac{1}{2}(\dim_R(X) - C_R(X) + 3) :$$

$\dim_R X$ denotes the top dimension n with $H^n(X, R) \neq 0$, and $C_R(X)$ is the smallest dimension $n \geq 1$ with $H^n(X, R) \neq 0$. The reason for the inequality in the theorem is that the homotopy group $\pi_m(S^n)$ of a sphere S^n has no p-torsion for $p < \frac{1}{2}(m - n + 3)$.

These results make it already sufficiently apparent that indeed we can exploit rational homotopy theory in the non-rational case. Unfortunately, a difficulty in the way of such an approach is that the methods of proof in rational homotopy theory are not at all available in the non-rational situation. Thus an entirely new approach is necessary.

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INTRODUCTION TO PART A

We obtain the results of part B by an extensive and systematic study of the algebraic properties of the classical homotopy operations

composition of maps $\circ$
smash products $\#$, $\underline{\#}$
Whitehead product $[\ ,\ ]$
James-Hopf invariants γ_n
addition $+$

It is much easier and of more general interest to consider these operations in their generalized form, namely

$$\circ : [\Sigma A, \Sigma B] \times [\Sigma B, Z] \to [\Sigma A, Z]$$

$$\#, \underline{\#} : [\Sigma X, \Sigma A] \times [\Sigma Y, \Sigma B] \to [\Sigma X \wedge Y, \Sigma A \wedge B]$$

$$[\ ,\ ] : [\Sigma A, Z] \times [\Sigma B, Z] \to [\Sigma A \wedge B, Z]$$

$$\gamma_n : [\Sigma A, \Sigma B] \to [\Sigma A, \Sigma B^{\wedge n}]$$

$$+ : [\Sigma A, Z] \times [\Sigma A, Z] \to [\Sigma A, Z] .$$

Many formulas relating to these operations are scattered through the literature. In the beginnings of homotopy theory the operations were only considered on homotopy groups $\pi_n(Y) = [S^n, Y]$. It took some time before the significance of the generalized operations became evident. The Whitehead product was invented by J. H. C. Whitehead in 1941. Arkowitz and Barratt obtained its generalization around 1960. In 1955 James gave his wonderful combinatorial definition of the higher-order Hopf invariants γ_n that is fundamental to our work. The nature of the higher invariants $(n > 2)$ remained unclear. They later were more systematically studied by Boardman and Steer (1967). However, their point of view is too stable-minded for our purposes, since they only consider the suspended invariants

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$$\lambda_n(\alpha) = \Sigma^{n-1}\gamma_n(\alpha) .$$

Moreover, they still use the left distributivity law for expanding the composite $(\xi + \eta)\alpha$ in terms of the Hilton-Hopf invariants as it was presented by Hilton in 1955. In this book we exhibit a more agreeable left distributivity law in terms of the James-Hopf invariants, namely

$$(*) \qquad \xi \circ \alpha + \eta \circ \alpha = (\xi + \eta) \circ \alpha + \sum_{n\geq 2} c_n(\xi, \eta) \circ \gamma_n(\alpha) .$$

(We use this formula to prove explicitly the folklore result that the James-Hopf invariants determine the Hilton-Hopf invariants.) A further major result of this book is the expansion formula for the Whitehead product $[\xi \circ \alpha, \eta \circ \beta]$ of composition elements $\xi \circ \alpha$ and $\eta \circ \beta$. This formula again uses the James-Hopf invariants and is of the form

$$(**) \qquad [\xi \circ \alpha, \eta \circ \beta] = \sum_{n\geq 1} \sum_{m\geq 1} R_{m,n}(\xi, \eta) \circ (\gamma_m(\alpha) \,\underline{\#}\, \gamma_n(\beta)) .$$

A special case of this formula was already found by Barcus-Barratt in 1958. The terms $c_n(\xi, \eta)$ and $R_{m,n}(\xi, \eta)$ are sums of iterated Whitehead products in ξ and η. We construct these terms explicitly. These two expansion formulas are basic to the development of our theory. Their proof makes use of classical commutator calculus in nilpotent group theory and Lie algebra theory. Chapter I therefore is purely algebraic. Various results of chapter I, while motivated by homotopy theory, seem to be new. They also may be of interest in combinatorial group theory and Lie algebra theory.

One of our crucial observations is that the above expansion formulas (*) and (**) are in fact closely connected with the following two formulas for the exponential function

$$e^x = \sum_{n\geq 0} x^n/n!$$

in a free tensor algebra. The Baker-Campbell-Hausdorff formula presents an infinite sum $\Phi(x, y)$ of <u>rational</u> Lie elements with the property

$$e^x e^y = e^{\Phi(x, y)} .$$

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