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# Variational Problems in Differential Geometry

University of Leeds 2009

*Edited by*

R. BIELAWSKI

K. HOUSTON

J.M. SPEIGHT

*University of Leeds*



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## Preface

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The workshop *Variational Problems in Differential Geometry* was held at the University of Leeds from March 30 to April 2nd, 2009.

The aim of the meeting was to bring together researchers working on disparate geometric problems, all of which admit a variational formulation. Among the topics discussed were recent developments in harmonic maps and morphisms, minimal and CMC surfaces, extremal Kähler metrics, the Yamabe functional, Hamiltonian variational problems, and topics related to gauge theory and to the Ricci flow.

The meeting incorporated a special session in honour of John C. Wood, on the occasion of his 60th birthday, to celebrate his seminal contributions to the theory of harmonic maps and morphisms.

The following mathematicians gave one-hour talks: Bernd Ammann, Claudio Arezzo, Paul Baird, Olivier Biquard, Christoph Boehm, Francis Burstall, Josef Dorfmeister, Akito Futaki, Mark Haskins, Frederic Helein, Nicolaos Kapouleas, Mario Micallef, Frank Pacard, Simon Salamon, Lorenz Schwachhoefer, Peter Topping, Richard Wentworth, and Jon Wolfson.

There were about 50 participants from the UK, US, Japan and several European countries. The schedule allowed plenty of opportunities for discussion and interaction between official talks and made for a successful and stimulating meeting.

The workshop was financially supported by the London Mathematical Society, the Engineering and Physical Sciences Research Council of Great Britain and the School of Mathematics, University of Leeds.

The articles presented in this volume represent the whole spectrum of the subject.

*The supremum of first eigenvalues of conformally covariant operators in a conformal class* by Ammann and Jammes is concerned with the first eigenvalues of the Yamabe operator, the Dirac operator, and more general conformally

covariant elliptic operators on compact Riemannian manifolds. It is well known that the infimum of the first eigenvalue in a given conformal class reflects a rich geometric structure. In this article, the authors study the supremum of the first eigenvalue and show that, for a very general class of operators, this supremum is infinite.

The article, *K-Destabilizing test configurations with smooth central fiber* by Arezzo, Della Vedova, and La Nave is concerned with the famous Tian-Yau-Donaldson conjecture about existence of constant scalar curvature Kähler metrics. They construct many new families of  $K$ -unstable manifolds, and, consequently, many new examples of manifolds which do not admit Kähler constant scalar curvature metrics in some cohomology classes.

As has been now understood, a very natural extension of Einstein metrics are the Ricci solitons. These are the subject of Paul Baird's article *Explicit constructions of Ricci solitons*, in which he does precisely that: he constructs many explicit examples, including some in the more exotic geometries  $Sol_3$ ,  $Nil_3$ , and  $Nil_4$ .

Josef Dorfmeister is concerned with a more classical topic: that of constant mean curvature and Willmore surfaces. In recent years, many new examples of such surfaces were constructed using loop groups. The method relies on finding "Iwasawa-like" decompositions of loop groups and the article *Open Iwasawa cells in twisted loop groups and some applications to harmonic maps* discusses such decompositions and their singularities.

The currently extremely important notions of  $K$ -stability and  $K$ -polystability are the topic of the paper by Futaki and Sano *Multiplier ideal sheaves and geometric problems*. This is an expository article giving state-of-the-art presentation of the powerful method of multiplier ideal sheaves and their applications to Kähler-Einstein and Sasaki-Einstein geometries.

*Multisymplectic formalism and the covariant phase space* by Frédéric Hélein takes us outside Riemannian geometry. The author presents an alternative (in fact, two of them) to the Feynman integral as a foundation of quantum field theory.

Lorenz Schwachhöfer's *Nonnegative curvature on disk bundles* is a survey of the glueing method used to construct Riemannian manifolds with nonnegative sectional curvature - one of the classical problems in geometry.

*Morse theory and stable pairs* by Wentworth and Wilkin introduces new techniques to compute equivariant cohomology of certain natural moduli spaces. The main ingredient is a version of Morse-Atiyah-Bott theory adapted to singular infinite dimensional spaces.

The final article, *Manifolds with  $k$ -positive Ricci curvature*, by Jon Wolfson, is a survey of results and conjectures about Riemannian  $n$ -manifolds with

$k$ -positive Ricci curvature. These interpolate between positive scalar curvature ( $n$ -positive Ricci curvature) and positive Ricci curvature (1-positive Ricci curvature), and the author shows how the results about  $k$ -positive Ricci curvature,  $1 < k < n$ , also interpolate, or should do, between what is known about manifolds satisfying those two classical notions of positivity.

We would like to extend our thanks to our colleague John Wood for his help and assistance in preparing these proceedings.

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