1

The supremum of first eigenvalues of conformally covariant operators in a conformal class

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Abstract

Let (M, g) be a compact Riemannian manifold of dimension ≥ 3. We show that there is a metric $\tilde{g}$ conformal to g and of volume 1 such that the first positive eigenvalue of the conformal Laplacian with respect to $\tilde{g}$ is arbitrarily large. A similar statement is proven for the first positive eigenvalue of the Dirac operator on a spin manifold of dimension ≥ 2.

1.1 Introduction

The goal of this article is to prove the following theorems.

Theorem 1.1.1 *Let (M, g_0, χ) be compact Riemannian spin manifold of dimension $n \geq 2$. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of the Dirac operator on (M, g, χ) by $\lambda_1^+(D_g)$. Then*

$$\sup_{g \in [g_0]} \lambda_1^+(D_g) \mathrm{Vol}(M, g)^{1/n} = \infty.$$

Theorem 1.1.2 *Let (M, g_0, χ) be compact Riemannian manifold of dimension $n \geq 3$. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of the conformal Laplacian $L_g := \Delta_g + \frac{n-2}{4(n-1)}\mathrm{Scal}_g$ (also called Yamabe operator) on (M, g, χ) by $\lambda_1^+(L_g)$. Then*

$$\sup_{g \in [g_0]} \lambda_1^+(L_g) \mathrm{Vol}(M, g)^{2/n} = \infty.$$

The Dirac operator and the conformal Laplacian belong to a large family of operators, defined in details in subsection 1.2.3. These operators are called conformally covariant elliptic operators of order k and of bidegree $((n-k)/2, (n+k)/2)$, acting on manifolds (M, g) of dimension $n > k$. In our definition we also claim formal self-adjointness.

All such conformally covariant elliptic operators of order k and of bidegree $((n-k)/2, (n+k)/2)$ share several analytical properties, in particular they are associated to the non-compact embedding $H^{k/2} \to L^{2n/(n-k)}$. Often they have interpretations in conformal geometry. To give an example, we define for a compact Riemannian manifold (M, g_0)

$$Y(M, [g_0]) := \inf_{g \in [g_0]} \lambda_1(L_g) \mathrm{Vol}(M, g)^{2/n},$$

where $\lambda_1(L_g)$ is the lowest eigenvalue of L_g. If $Y(M, [g_0]) > 0$, then the solution of the Yamabe problem [29] tells us that the infimum is attained and the minimizer is a metric of constant scalar curvature. This famous problem was finally solved by Schoen and Yau using the positive mass theorem.

In a similar way, for $n = 2$ the Dirac operator is associated to constant-mean-curvature conformal immersions of the universal covering into $\mathbb{R}^3$. If a Dirac-operator-analogue of the positive mass theorem holds for a given manifold (M, g_0), then the infimum

$$\inf_{g \in [g_0]} \lambda_1^+(D_g) \mathrm{Vol}(M, g)^{1/n}$$

is attained [3]. However, it is still unclear whether such a Dirac-operator-analogue of the positive mass theorem holds in general.

The Yamabe problem and its Dirac operator analogue, as well as the analogues for other conformally covariant operators are typically solved by minimizing an associated variational problem. As the Sobolev embedding $H^{k/2} \to L^{2n/(n-k)}$ is non-compact, the direct method of the calculus of variation fails, but perturbation techniques and conformal blow-up techniques typically work. Hence all these operators share many properties.

However, only few statements can be proven simultaneously for all conformally covariant elliptic operators of order k and of bidegree $((n-k)/2, (n+k)/2)$. Some of the operators are bounded from below (e.g. the Yamabe and the Paneitz operator), whereas others are not (e.g. the Dirac operator). Some of them admit a maximum principle, others do not. Some of them act on functions, others on sections of vector bundles. The associated Sobolev space $H^{k/2}$ has non-integer order if k is odd, hence it is not the natural domain of a differential operator. For Dirac operators, the spin structure has to be considered in order to derive a statement as Theorem 1.1.1 for $n = 2$. Because of these differences, most analytical properties have to be proven for each operator separately.

We consider it hence as remarkable that the proof of our Theorems 1.1.1 and 1.1.2 can be extended to all such operators. Our proof only uses some few properties of the operators, defined axiomatically in 1.2.3. More exactly we prove the following.

Theorem 1.1.3 *Let P_g be a conformally covariant elliptic operator of order k, of bidegree $((n-k)/2, (n+k)/2)$ acting on manifolds of dimension $n > k$. We also assume that P_g is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$ (see Definition 1.2.4). Let (M, g_0) be compact Riemannian manifold. In the case that P_g depends on the spin structure, we assume that M is oriented and is equipped with a spin structure. For any metric g in the conformal class $[g_0]$, we denote the first positive eigenvalue of P_g by $\lambda_1^+(P_g)$. Then*

$$\sup_{g \in [g_0]} \lambda_1^+(P_g) \mathrm{Vol}(M, g)^{k/n} = \infty.$$

The interest in this result is motivated by three questions. At first, as already mentioned above the infimum

$$\inf_{g \in [g_0]} \lambda_1^+(D_g) \mathrm{Vol}(M, g)^{1/n}$$

reflects a rich geometrical structure [3], [4], [5], [7], [8], similarly for the conformal Laplacian. It seems natural to study the supremum as well.

The second motivation comes from comparing Theorem 1.1.3 to results about some other differential operators. For the Hodge Laplacian Δ_p^g acting on p-forms, we have $\sup_{g \in [g_0]} \lambda_1(\Delta_p^g) \mathrm{Vol}(M, g)^{2/n} = +\infty$ for $n \geq 4$ and $2 \leq p \leq n-2$ ([19]). On the other hand, for the Laplacian Δ^g acting on functions, we have

$$\sup_{g \in [g_0]} \lambda_k(\Delta^g) \mathrm{Vol}(M, g)^{2/n} < +\infty$$

(the case $k = 1$ is proven in [20] and the general case in [27]). See [25] for a synthetic presentation of this subject.

The essential idea in our proof is to construct metrics with longer and longer cylindrical parts. We will call this an *asymptotically cylindrical blowup*. Such metrics are also called *Pinocchio metrics* in [2, 6]. In [2, 6] the behavior of Dirac eigenvalues on such metrics has already been studied partially, but the present article has much stronger results. To extend these existing results provides the third motivation.

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1.2 Preliminaries

1.2.1 Notations

In this article $B_y(r)$ denotes the ball of radius r around y, $S_y(r) = \partial B_y(r)$ its boundary. The standard sphere $S_0(1) \subset \mathbb{R}^n$ in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$, its volume is ω_{n-1}. For the volume element of (M, g) we use the notation dv^g. In our article, $\Gamma(V)$ (resp. $\Gamma_c(V)$) always denotes the set of all smooth sections (resp. all compactly supported smooth sections) of the vector bundle $V \to M$.

For sections u of $V \to M$ over a Riemannian manifold (M, g) the Sobolev norms L^2 and H^s, $s \in \mathbb{N}$, are defined as

$$\|u\|^2_{L^2(M,g)} := \int_M |u|^2 \, dv^g$$
$$\|u\|^2_{H^s(M,g)} := \|u\|^2_{L^2(M,g)} + \|\nabla u\|^2_{L^2(M,g)} + \cdots + \|\nabla^s u\|^2_{L^2(M,g)}.$$

The vector bundle V will be suppressed in the notation. If M and g are clear from the context, we write just L^2 and H^s. The completion of $\{u \in \Gamma(V) \mid \|u\|_{H^s(M,g)} < \infty\}$ with respect to the $H^s(M, g)$-norm is denoted by $\Gamma_{H^s(M,g)}(V)$, or if (M, g) or V is clear from the context, we alternatively write $\Gamma_{H^s}(V)$ or $H^s(M, g)$ for $\Gamma_{H^s(M,g)}(V)$. The same definitions are used for L^2 instead of H^s. And similarly $\Gamma_{C^k(M,g)}(V) = \Gamma_{C^k}(V) = C^k(M, g)$ is the set of all C^k-sections, $k \in \mathbb{N} \cup \{\infty\}$.

1.2.2 Removal of singularities

In the proof we will use the following removal of singularities lemma.

Lemma 1.2.1 (Removal of singularities lemma) *Let Ω be a bounded open subset of $\mathbb{R}^n$ containing 0. Let P be an elliptic differential operator of order k on Ω, $f \in C^\infty(\Omega)$, and let $u \in C^\infty(\Omega \setminus \{0\})$ be a solution of*

$$Pu = f \tag{1.1}$$

on $\Omega \setminus \{0\}$ with

$$\lim_{\varepsilon \to 0} \int_{B_0(2\varepsilon) - B_0(\varepsilon)} |u| r^{-k} = 0 \ \text{ and } \ \lim_{\varepsilon \to 0} \int_{B_0(\varepsilon)} |u| = 0 \tag{1.2}$$

where r is the distance to 0. Then u is a (strong) solution of (1.1) on Ω. The same result holds for sections of vector bundles over relatively compact open subset of Riemannian manifolds.

Proof We show that u is a weak solution of (1.1) in the distributional sense, and then it follows from standard regularity theory, that it is also a strong solution. This means that we have to show that for any given compactly supported smooth test function $\psi : \Omega \to \mathbb{R}$ we have

$$\int_\Omega u P^* \psi = \int_\Omega f \psi.$$

Let $\eta : \Omega \to [0, 1]$ be a test function that is identically 1 on $B_0(\varepsilon)$, has support in $B_0(2\varepsilon)$, and with $|\nabla^m \eta| \le C_m/\varepsilon^m$. It follows that

$$\sup |P^*(\eta\psi)| \le C(P, \Omega, \psi)\varepsilon^{-k},$$

on $B_0(2\varepsilon) \setminus B_0(\varepsilon)$ and $\sup |P^*(\eta\psi)| \le C(P, \Omega, \psi)$ on $B_0(\varepsilon)$ and hence

$$\begin{aligned} \left| \int_\Omega u P^*(\eta\psi) \right| &\le C\varepsilon^{-k} \int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| + C \int_{B_0(\varepsilon)} |u| \\ &\le C \int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| r^{-k} + C \int_{B_0(\varepsilon)} |u| \to 0. \end{aligned} \tag{1.3}$$

We conclude

$$\begin{aligned} \int_\Omega u P^* \psi &= \int_\Omega u P^*(\eta\psi) + \int_\Omega u P^*((1-\eta)\psi) \\ &= \underbrace{\int_\Omega u P^*(\eta\psi)}_{\to 0} + \underbrace{\int_\Omega (Pu)(1-\eta)\psi}_{\to \int_\Omega f\psi} \end{aligned} \tag{1.4}$$

for $\varepsilon \to 0$. Hence the lemma follows. □

Condition (1.2) is obviously satisfied if $\int_\Omega |u| r^{-k} < \infty$. It is also satisfied if

$$\int_\Omega |u|^2 r^{-k} < \infty \text{ and } k \le n, \tag{1.5}$$

as in this case

$$\left(\int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} |u| r^{-k} \right)^2 \le \int_\Omega |u|^2 r^{-k} \underbrace{\int_{B_0(2\varepsilon)\setminus B_0(\varepsilon)} r^{-k}}_{\le C}.$$

1.2.3 Conformally covariant elliptic operators

In this subsection we present a class of certain conformally covariant elliptic operators. Many important geometric operators are in this class, in particular the conformal Laplacian, the Paneitz operator, the Dirac operator, see also [18, 21, 22] for more examples. Readers who are only interested in the Dirac operator, the Conformal Laplacian or the Paneitz operator, can skip this part and continue with section 1.3.

Such a conformally covariant operator is not just one single differential operator, but a procedure how to associate to an n-dimensional Riemannian manifold (M, g) (potentially with some additional structure) a differential operator P_g of order k acting on a vector bundle. The important fact is that if $g_2 = f^2 g_1$, then one claims

$$P_{g_2} = f^{-\frac{n+k}{2}} P_{g_1} f^{\frac{n-k}{2}}. \tag{1.6}$$

One also expresses this by saying that P has bidegree $((n-k)/2, (n+k)/2)$.

The sense of this equation is apparent if P_g is an operator from $C^\infty(M)$ to $C^\infty(M)$. If P_g acts on a vector bundle or if some additional structure (as e.g. spin structure) is used for defining it, then a rigorous and careful definition needs more attention. The language of categories provides a good formal framework [30]. The concept of conformally covariant elliptic operators is already used by many authors, but we do not know of a reference where a formal definition is carried out that fits to our context. (See [26] for a similar categorial approach that includes some of the operators presented here.) Often an intuitive definition is used. The intuitive definition is obviously sufficient if one deals with operators acting on functions, such as the conformal Laplacian or the Paneitz operator. However to properly state Theorem 1.1.3 we need the following definition.

Let $Riem^n$ (resp. $Riemspin^n$) be the category n-dimensional Riemannian manifolds (resp. n-dimensional Riemannian manifolds with orientation and spin structure). Morphisms from (M_1, g_1) to (M_2, g_2) are conformal embeddings $(M_1, g_1) \hookrightarrow (M_2, g_2)$ (resp. conformal embeddings preserving orientation and spin structure).

Let $Laplace^n_k$ (resp. $Dirac^n_k$) be the category whose objects are

$$\{(M, g), V_g, P_g\}$$

where (M, g) in an object of $Riem^n$ (resp. $Riemspin^n$), where V_g is a vector bundle with a scalar product on the fibers, where $P_g : \Gamma(V_g) \to \Gamma(V_g)$ is an elliptic formally self-adjoint differential operator of order k.

A morphism (ι, κ) from $\{(M_1, g_1), V_{g_1}, P_{g_1}\}$ to $\{(M_2, g_2), V_{g_2}, P_{g_2}\}$ consists of a conformal embedding $\iota : (M_1, g_1) \hookrightarrow (M_2, g_2)$ (preserving orientation and spin structure in the case of $Dirac^n_k$) together with a fiber isomorphism $\kappa : \iota^* V_{g_2} \to V_{g_1}$ preserving fiberwise length, such that P_{g_1} and P_{g_2} satisfy the conformal covariance property (1.6). For stating this property precisely, let $f > 0$ be defined by $\iota^* g_2 = f^2 g_1$, and let $\kappa_* : \Gamma(V_{g_2}) \to \Gamma(V_{g_1})$, $\kappa_*(\varphi) = \kappa \circ \varphi \circ \iota$. Then the conformal covariance property is

$$\kappa_* P_{g_2} = f^{-\frac{n+k}{2}} P_{g_1} f^{\frac{n-k}{2}} \kappa_*. \tag{1.7}$$

In the following the maps κ and ι will often be evident from the context and then will be omitted. The transformation formula (1.7) then simplifies to (1.6).

Definition 1.2.2 A *conformally covariant elliptic operator of order k and of bidegree* $((n-k)/2, (n+k)/2)$ is a contravariant functor from *Riem*^{n} (resp. *Riemspin*^{n}) to *Laplace*^n_k (resp. *Dirac*^n_k), mapping (M, g) to (M, g, V_g, P_g) in such a way that the coefficients are continuous in the C^k-topology of metrics (see below). To shorten notation, we just write P_g or P for this functor.

It remains to explain the C^k-continuity of the coefficients.

For Riemannian metrics g, g_1, g_2 defined on a compact set $K \subset M$ we set

$$d^g_{C^k(K)}(g_1, g_2) := \max_{t=0,\ldots,k} \|(\nabla_g)^t (g_1 - g_2)\|_{C^0(K)}.$$

For a fixed background metric g, the relation $d^g_{C^k(K)}(\cdot\,,\,\cdot)$ defines a distance function on the space of metrics on K. The topology induced by d^g is independent of this background metric and it is called the *C^k-topology of metrics on K*.

Definition 1.2.3 We say that the coefficients of P are *continuous in the C^k-topology of metrics* if for any metric g on a manifold M, and for any compact subset $K \subset M$ there is a neighborhood $\mathcal{U}$ of $g|_K$ in the C^k-topology of metrics on K, such that for all metrics $\tilde{g}$, $\tilde{g}|_K \in \mathcal{U}$, there is an isomorphism of vector bundles $\hat{\kappa} : V_g|_K \to V_{\tilde{g}}|_K$ over the identity of K with induced map $\hat{\kappa}_* : \Gamma(V_g|_K) \to \Gamma(V_{\tilde{g}}|_K)$ with the property that the coefficients of the differential operator

$$P_g - (\hat{\kappa}_*)^{-1} P_{\tilde{g}} \hat{\kappa}_*$$

depend continuously on $\tilde{g}$ (with respect to the C^k-topology of metrics).

1.2.4 Invertibility on $\mathbb{S}^{n-1} \times \mathbb{R}$

Let P be a conformally covariant elliptic operator of order k and of bidegree $((n-k)/2, (n+k)/2)$. For $(M, g) = \mathbb{S}^{n-1} \times \mathbb{R}$, the operator P_g is a self-adjoint operator $H^k \subset L^2 \to L^2$ (see Lemma 1.3.1 and the comments thereafter).

Definition 1.2.4 We say that P *is invertible on* $\mathbb{S}^{n-1} \times \mathbb{R}$ if P_g is an invertible operator $H^k \to L^2$ where g is the standard product metric on $\mathbb{S}^{n-1} \times \mathbb{R}$. In order words there is a constant $\sigma > 0$ such that the spectrum of $P_g : \Gamma_{H^k}(V_g) \to \Gamma_{L^2}(V_g)$ is contained in $(-\infty, -\sigma] \cup [\sigma, \infty)$ for any $g \in U$. In the following, the largest such σ will be called σ_P.

We conjecture that any conformally covariant elliptic operator of order k and of bidegree $((n-k)/2, (n+k)/2)$ with $k < n$ is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$.

1.2.5 Examples

Example 1: The Conformal Laplacian
Let

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} \mathrm{Scal}_g,$$

be the conformal Laplacian. It acts on functions on a Riemannian manifold (M, g), i.e. V_g is the trivial real line bundle $\underline{\mathbb{R}}$. Let $\iota : (M_1, g_1) \hookrightarrow (M_2, g_2)$ be a conformal embedding. Then we can choose $\kappa := \mathrm{Id} : \iota^* V_{g_2} \to V_{g_1}$ and formula (1.7) holds for $k = 2$ (see e.g. [15, Section 1.J]). All coefficients of L_g depend continuously on g in the C^2-topology. Hence L is a conformally covariant elliptic operator of order 2 and of bidegree $((n-2)/2, (n+2)/2)$.

The scalar curvature of $\mathbb{S}^{n-1} \times \mathbb{R}$ is $(n-1)(n-2)$. The spectrum of L_g on $\mathbb{S}^{n-1} \times \mathbb{R}$ of L_g coincides with the essential spectrum of L_g and is $[\sigma_L, \infty)$ with $\sigma_L := (n-2)^2/4$. Hence L is invertible on $S^{n-1} \times \mathbb{R}$ if (and only if) $n > 2$.

Example 2: The Paneitz operator

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 5$. The Paneitz operator P_g is given by

$$P_g u = (\Delta_g)^2 u - \mathrm{div}_g(A_g\, du) + \frac{n-4}{2} Q_g u$$

where

$$A_g := \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \mathrm{Scal}_g g - \frac{4}{n-2} \mathrm{Ric}_g,$$

$$Q_g = \frac{1}{2(n-1)} \Delta_g \mathrm{Scal}_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \mathrm{Scal}_g^2 - \frac{2}{(n-2)^2} |\mathrm{Ric}_g|^2.$$

This operator was defined by Paneitz [32] in the case $n = 4$, and it was generalized by Branson in [17] to arbitrary dimensions ≥ 4. We also refer to Theorem 1.21 of the overview article [16]. The explicit formula presented above can be found e.g. in [23]. The coefficients of P_g depend continuously on g in the C^4-topology

As in the previous example we can choose for κ the identity, and then the Paneitz operator P_g is a conformally covariant elliptic operator of order 4 and of bidegree $((n-4)/2, (n+4)/2)$.

On $\mathbb{S}^{n-1} \times \mathbb{R}$ one calculates

$$A_g = \frac{(n-4)n}{2} \mathrm{Id} + 4\pi_{\mathbb{R}} > 0$$

where $\pi_{\mathbb{R}}$ is the projection to vectors parallel to $\mathbb{R}$.

$$Q_g = \frac{(n-4)n^2}{8}.$$

We conclude

$$\sigma_P = Q = \frac{(n-4)n^2}{8}$$

and P is invertible on $\mathbb{S}^{n-1} \times \mathbb{R}$ if (and only if) $n > 4$.

Examples 3: The Dirac operator.

Let $\tilde{g} = f^2 g$. Let $\Sigma_g M$ resp. $\Sigma_{\tilde{g}} M$ be the spinor bundle of (M, g) resp. $(M, \tilde{g})$. Then there is a fiberwise isomorphism $\beta^g_{\tilde{g}} : \Sigma_g M \to \Sigma_{\tilde{g}} M$, preserving the norm such that

$$D_{\tilde{g}} \circ \beta^g_{\tilde{g}}(\varphi) = f^{-\frac{n+1}{2}} \beta^g_{\tilde{g}} \circ D_g \left(f^{\frac{n-1}{2}} \varphi \right),$$

see [24, 14] for details. Furthermore, the cocycle conditions

$$\beta^g_{\tilde{g}} \circ \beta^{\tilde{g}}_g = \mathrm{Id} \qquad \text{and} \qquad \beta^{\hat{g}}_g \circ \beta^{\tilde{g}}_{\hat{g}} \circ \beta^g_{\tilde{g}} = \mathrm{Id}$$

hold for conformal metrics g, $\tilde{g}$ and $\hat{g}$. We will hence use the map $\beta^g_{\tilde{g}}$ to identify $\Sigma_g M$ with $\Sigma_{\tilde{g}} M$. Hence we simply get

$$D_{\tilde{g}} \varphi = f^{-\frac{n+1}{2}} \circ D_g \left(f^{\frac{n-1}{2}} \varphi \right). \tag{1.8}$$

All coefficients of D_g depend continuously on g in the C^1-topology. Hence D is a conformally covariant elliptic operator of order 1 and of bidegree $((n-1)/2, (n+1)/2)$.

The Dirac operator on $\mathbb{S}^{n-1} \times \mathbb{R}$ can be decomposed in a part D_{vert} deriving along $\mathbb{S}^{n-1}$ and a part D_{hor} deriving along $\mathbb{R}$, $D_g = D_{\mathrm{vert}} + D_{\mathrm{hor}}$, see [1] or [2].

Locally

$$D_{\text{vert}} = \sum_{i=1}^{n-1} e_i \cdot \nabla_{e_i}$$

for a local frame $(e_1, \ldots, e_{n-1})$ of $\mathbb{S}^{n-1}$. Here $\cdot$ denotes the Clifford multiplication $TM \otimes \Sigma_g M \to \Sigma_g M$. Furthermore $D_{\text{hor}} = \partial_t \cdot \nabla_{\partial_t}$, where $t \in \mathbb{R}$ is the standard coordinate of $\mathbb{R}$. The operators D_{vert} and D_{hor} anticommute. For $n \geq 3$, the spectrum of D_{vert} coincides with the spectrum of the Dirac operator on $\mathbb{S}^{n-1}$, we cite [12] and obtain

$$\operatorname{spec} D_{\text{vert}} = \left\{ \pm \left(\frac{n-1}{2} + k \right) \mid k \in \mathbb{N}_0 \right\}.$$

The operator $(D_{\text{hor}})^2$ is the ordinary Laplacian on $\mathbb{R}$ and hence has spectrum $[0, \infty)$. Together this implies that the spectrum of the Dirac operator on $\mathbb{S}^{n-1} \times \mathbb{R}$ is the set $(-\infty, -\sigma_D] \cup [\sigma_D, \infty)$ with $\sigma_D = \frac{n-1}{2}$.

In the case $n = 2$ these statements are only correct if the circle $\mathbb{S}^{n-1} = \mathbb{S}^1$ carries the spin structure induced from the ball. Only this spin structure extends to the conformal compactification that is given by adding one point at infinity for each end. For this reason, we will understand in the whole article that all circles $\mathbb{S}^1$ should be equipped with this bounding spin structure. The extension of the spin structure is essential in order to have a spinor bundle on the compactification. The methods used in our proof use this extension implicitly.

Hence D is invertible on $S^{n-1} \times \mathbb{R}$ if (and only if) $n > 1$.

Most techniques used in the literature on estimating eigenvalues of the Dirac operators do not use the spin structure and hence these techniques cannot provide a proof in the case $n = 2$.

Example 4: The Rarita-Schwinger operator and many other Fegan type operators are conformally covariant elliptic operators of order 1 and of bidegree $((n-1)/2, (n+1)/2)$. See [21] and in the work of T. Branson for more information.

Example 5: Assume that (M, g) is a Riemannian spin manifold that carries a vector bundle $W \to M$ with metric and metric connection. Then there is a natural first order operator $\Gamma(\Sigma_g M \otimes W) \to \Gamma(\Sigma_g M \otimes W)$, the *Dirac operator twisted by* W. This operator has similar properties as conformally covariant elliptic operators of order 1 and of bidegree $((n-1)/2, (n+1)/2)$. The methods of our article can be easily adapted in order to show that Theorem 1.1.3 is also true for this twisted Dirac operator. However, twisted Dirac operators are not "conformally covariant elliptic operators" in the above sense. They could have been included in this class by replacing the category *Riemspin*^{n} by