Cambridge University Press & Assessment
978-0-521-28235-2 — Circuit Double Cover of Graphs
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Excerpt
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Circuit double cover

Most terminology and notation in this book follow standard textbooks in graph theory (such as [18], [19], [41], [242], etc.). Some terminology is slightly different from those classical textbooks.

Definition 1.0.1 Let $G = (V, E)$ be a graph with vertex set V and edge set E.

(1) A circuit is a connected 2-regular graph.

(2) A graph (subgraph) is even if the degree of each vertex is even.

(3) A bridge (or cut-edge) of a graph G is an edge whose removal increases the number of components of G (equivalently, a bridge is an edge that is not contained in any circuit of G).

Note that, in much of the literature related to circuit covers and integer flows, an even graph/subgraph is also called a cycle, which is adapted from matroid theory, and is different from what is used in many other graph theory textbooks. For the sake of less confusion, we will use *even graph/subgraph* instead of *cycle* in this book.

Definition 1.0.2 (1) A family $\mathcal{F}$ of circuits (or even subgraphs) of a graph G is called a circuit cover (or an even subgraph cover) if every edge of G is contained in some member(s) of $\mathcal{F}$.

(2) A circuit cover (or an even subgraph cover) $\mathcal{F}$ of a graph G is a double cover of G if every edge is contained in precisely two members of $\mathcal{F}$.

Graphs considered in this book may contain loops or parallel edges. However, most graphs are bridgeless (since this is necessary for circuit covering problems).

1.1 Circuit double cover conjecture

The following is one of the most well-known open problems in graph theory, and is the major subject of this book.

The Circuit Double Cover (CDC) Conjecture. *Every bridgeless graph G has a family $\mathcal{F}$ of circuits such that every edge of G is contained in precisely two members of $\mathcal{F}$.*

The CDC conjecture was presented as an "open question" by Szekeres ([219] p. 374) for cubic graphs (as we will see soon in Section 1.2, it is equivalent for all bridgeless graphs). The conjecture was also independently stated by Seymour in [205] (Conjecture 3.3 on p. 347) for all bridgeless graphs. An equivalent version of the CDC conjecture was proposed by Itai and Rodeh (Open problem (ii) in [119] p. 298) that *every bridgeless graph has a family $\mathcal{F}$ of circuits such that every edge is contained in one or two members of $\mathcal{F}$.* (It is not hard to see that this open problem is equivalent to the circuit double cover conjecture. See Exercise 1.3.)

For the origin of the conjecture, some mathematicians gave the credit to Tutte. According to a personal letter from Tutte to Fleischner [237], he said, *"I too have been puzzled to find an original reference. I think the conjecture is one that was well established in mathematical conversation long before anyone thought of publishing it."* It was also pointed out in the survey paper by Jaeger [130] that *"it seems difficult to attribute the paternity of this conjecture;"* and also pointed out in some early literatures (such as [82]) that *"its origin is uncertain."* This may explain why the CDC conjecture is considered as "folklore" in [19] (Unsolved problem 10, p. 584).

An early work related to circuit double cover that we are able to find is a paper by Tutte published in 1949 [229], [237].

The circuit double cover conjecture is obviously true for 2-connected planar graphs since the boundary of every face is a circuit and the set of the boundaries of all faces forms a circuit double cover of an embedded graph. One might attempt to extend this observation further to all 2-connected graphs embedded on some surfaces. However, it is not true that *any embedding* of a 2-connected graph is free of a handle-bridge. That is, the boundary of some face may not be a circuit. So, *can we find an embedding of a 2-connected graph G on some surface Σ such that the boundary of every face is a circuit?* The existence of such an embedding is an even stronger open problem in topology (see Section 1.4 for more detail).

The circuit double cover conjecture is also true for the family of 3-edge-colorable cubic graphs since a 3-edge-colorable cubic graph is covered by three bi-colored 2-factors (see Section 1.3). This observation will be further extended to all bridgeless graphs (not necessarily cubic) that admit nowhere-zero 4-flows (see Chapter 7 and Appendix C).

1.2 Minimal counterexamples

In this section, we study some basic structures of a smallest counterexample to the circuit double cover conjecture.

Definition 1.2.1 Let G be a graph. The suppressed graph of G is the graph obtained from G by replacing each maximal subdivided edge with a single edge, and is denoted by $\overline{G}$.

Lemma 1.2.2 [205] *Let G be a bridgeless graph. If G has no circuit double cover, then there is a bridgeless cubic graph G' such that $|E(G')| \leq |E(G)|$ and G' has no circuit double cover either.*

Proof Let $v \in V(G)$ with $d(v) \geq 4$. By Theorem A.1.14 (the vertex splitting method), there is a pair of edges e_1, e_2 of G incident with v such that the graph $G_{[v;\{e_1,e_2\}]}$ obtained from G by splitting e_1 and e_2 away from v remains bridgeless. Let G' be the suppressed graph of $G_{[v;\{e_1,e_2\}]}$. It is evident that any circuit double cover of G' can be adjusted or modified to a circuit double cover of G. Thus, G' does not have a circuit double cover. Repeating this procedure, we obtain a bridgeless cubic graph G'' which is smaller than G and has no circuit double cover either. □

Lemma 1.2.3 [205] *Let G be a bridgeless graph. If G has no circuit double cover, then G is contractible to an essentially 4-edge-connected graph G' such that G' has no circuit double cover either.*

Proof Let T be a non-trivial 2- or 3-edge-cut of G with components Q_1, Q_2. Let G_i be the graph obtained from G by contracting Q_i, for each $i = 1, 2$. If G_i has a circuit double cover $\mathcal{F}_i$ for each $i = 1, 2$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ can be modified at edges of T so that the resulting family of circuits is a circuit double cover of G (see Figure 1.1). □

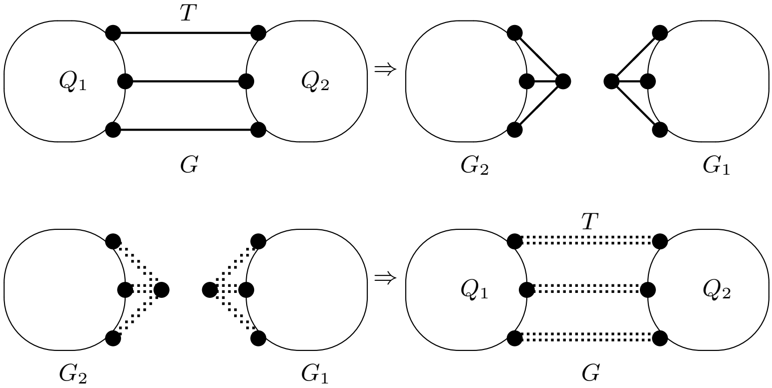

Figure 1.1 *CDC of G constructed from G_1 and G_2*

With some straightforward observations, the following theorem summarizes some structure of a minimal counterexample to the conjecture of circuit double cover.

Theorem 1.2.4 *If G is a minimum counterexample to the circuit double cover conjecture, then*

(1) G is simple, 3-connected and cubic (Lemmas 1.2.2 and 1.2.3);
(2) G has no non-trivial 2 or 3-edge-cut (Lemma 1.2.3);
(3) G is not 3-edge-colorable (see Section 1.3);
(4) G is not planar (see Sections 1.1 and 1.4).

There are more structural properties for a smallest counterexample to the conjecture. The following is a partial summary of some results we will discuss in this book.

(1) G contains no Hamilton path (Corollary 4.2.8).
(2) G contains a subdivision of the Petersen graph (Theorem 3.2.1).
(3) The girth of G is at least 12 (Theorem 9.1.1).
(4) For each edge $e \in E(G)$, the suppressed cubic graph $\overline{G - \{e\}}$ is not 3-edge-colorable (Theorem 2.2.4).
(5) $G \neq \overline{G' - \{e\}}$ for some 3-edge-colorable cubic graph G' and some $e \in E(G')$ (Theorem 4.2.3).
(6) G is of oddness at least 6 (Theorem 4.2.4).

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1.3 3-edge-coloring and even subgraph cover

Recall some definitions. A subgraph H is even if the degree of every vertex is even (defined in Definition 1.0.1). A family $\mathcal{F}$ of even subgraphs is called an even-subgraph double cover of a graph G if every edge e of G is contained in precisely 2 members of $\mathcal{F}$ (defined in Definition 1.0.2).

Definition 1.3.1 An even-subgraph double cover $\mathcal{F}$ of a graph G is called a k-even-subgraph double cover if $|\mathcal{F}| \leq k$.

It is trivial that every circuit double cover is an even subgraph double cover. And, by Lemma A.2.2, it is also straightforward that every even subgraph double cover can be converted to a circuit double cover (by replacing each even subgraph with a set of circuits).

The following theorem was formulated by Jaeger [130]. (The equivalence of (1) and (2) was also applied in [205].)

Theorem 1.3.2 *Let G be a cubic graph. Then the following statements are equivalent:*

(1) G is 3-edge-colorable;

(2) G has a 3-even subgraph double cover.

(3) G has a 4-even subgraph double cover.

Proof (1) $\Rightarrow$ (2): Let $c : E(G) \to \mathbb{Z}_3$ be a 3-edge-coloring of G. Then

$$\{c^{-1}(\alpha) \cup c^{-1}(\beta) : \ \alpha, \beta \in \mathbb{Z}_3, \alpha \neq \beta\}$$

is a 3-even subgraph double cover of G.

(2) $\Rightarrow$ (1): Let $\{C_\mu : \ \mu \in \mathbb{Z}_2 \times \mathbb{Z}_2 - \{(0,0)\}\}$ be a 3-even subgraph double cover of G. Then let $c : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 - \{(0,0)\}$ be defined as follows: for each edge $e \in E(G)$, $c(e) = \mu' + \mu''$ if e is covered by even subgraphs $C_{\mu'}$ and $C_{\mu''}$. It is easy to see that c is a proper 3-edge-coloring of G.

(2) $\Rightarrow$ (3): Trivial, let $C_4 = \emptyset$.

(3) $\Rightarrow$ (2): Let $\{C_1, C_2, C_3, C_4\}$ be a 4-even subgraph double cover of G. Then $\{C_1 \triangle C_2, C_1 \triangle C_3, C_1 \triangle C_4\}$ is a 3-even subgraph double cover of G. □

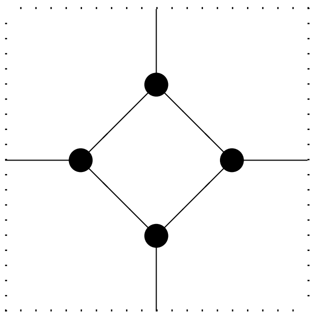

Figure 1.2 *A 2-cell embedding of K_4 on the torus that is not circular*

1.4 Circuit double covers and graph embeddings

All surfaces considered in this book are connected 2-manifolds (see [243] or [182] for an introduction to the topology of surfaces). The set of faces of an embedded graph G is denoted by $F(G)$.

Definition 1.4.1 Let G be a 2-connected graph and Σ be a surface. An embedding of G on Σ is called a 2-cell embedding if each face of the embedded graph is homeomorphic to the open unit disk.

Definition 1.4.2 Let G be a graph G that has a 2-cell embedding on a surface Σ.

(1) A coloring of the faces of the embedded graph G is proper if for each edge $e \in E(G)$, the faces on the two sides of e are colored differently.

(2) The embedded graph G is k-face-colorable on Σ if there is a proper face coloring of G requiring at most k colors.

Note that a 2-cell embedding of a graph G on a 2-manifold does not guarantee a proper face coloring of the graph G since some edge might be on the boundary of the same face (see Figure 1.2).

Thus, we have to consider an embedding with the following property.

Definition 1.4.3 A 2-cell embedding of a 2-connected graph G on a surface is circular if the boundary of each face is a circuit.

It is not hard to see that the definition of a circular 2-cell embedding of a graph on a surface has the following equivalent statements:

(1) each edge is on the boundaries of two distinct faces;

(2) the closure of each face is homeomorphic to the closed unit disk.

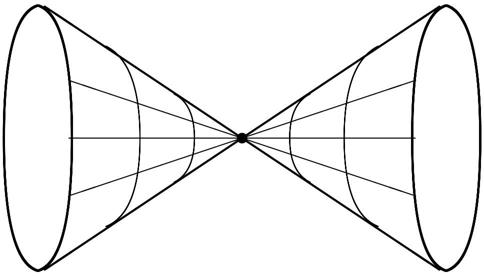

Figure 1.3 *A pinch point on a pseudo-surface*

For a 2-connected planar graph, the circuit double cover conjecture is obviously true since the family of all facial boundaries is a circuit double cover of the graph. This is also true for every 2-connected graph which has a circular 2-cell embedding on some surface. However, one cannot assume that every 2-connected graph has such an embedding, since that is an even stronger open problem.

Conjecture 1.4.4 *Every 2-connected graph has a circular 2-cell embedding on some 2-manifold.*

Conjecture 1.4.4 has also been considered as a "folklore" conjecture (see [185]) with possible origin due to Tutte in the mid-1960s (see [19], Conjecture 3.10 on p. 95, or [33]). It is also presented in the articles [97], [167] as an open question.

Note that the CDC conjecture and Conjecture 1.4.4 are equivalent for connected *cubic* graphs (Exercise 1.4). Let $\mathcal{F}$ be a circuit double cover of G. One can consider each circuit C of $\mathcal{F}$ as the boundary of a disk and join those disks at the edges of G; the resulting surface is certainly a 2-manifold.

For a non-cubic graph G, with the method we used in the above paragraph, the "surface" that we obtained from a circuit double cover may not be a 2-manifold. It is possible that some "pinch point" appears on the constructed "surface," which is sometime called a pseudo-surface (see Figure 1.3).

Some mathematicians believe that the circular 2-cell embedding is a feasible approach to the CDC conjecture, although Conjecture 1.4.4 is stronger. However, there is little progress yet in this direction.

In this book, we are mostly interested in the combinatorial structure of graphs, while the topological properties of graphs are not a major subject. Thus, this book is not designed to be a comprehensive study of graph embeddings. Some results on embedding are mentioned simply because they are corollaries or applications of combinatorics results (for example, Sections 9.2, 10.8, 13.1, etc.).

Partial results related to the circular 2-cell embedding conjecture can be found in articles, such as [47], [97], [167], [181], [218], [251], [252], [253], etc.

1.5 Open problems

Conjecture 1.5.1 (Strong CDC conjecture; Seymour, see [61] p. 237, also [62], [83]) *Let G be a bridgeless cubic graph and C be a circuit of G. The graph G has a circuit double cover $\mathcal{F}$ with $C \in \mathcal{F}$.*

A recent computer aided search [21] showed that Conjecture 1.5.1 holds for all cubic graphs of order at most 36.

Conjecture 1.5.2 (Sabidussi and Fleischner [60], and Conjecture 2.4 in [2] p. 462) *Let G be a cubic graph such that G has a dominating circuit C. Then G has a circuit double cover $\mathcal{F}$ such that C is a member of $\mathcal{F}$.*

Remark. (1) Conjecture 1.5.1 implies both the circuit double cover conjecture and Conjecture 1.5.2.

(2) Conjecture 1.5.2 is an equivalent version of the original Sabidussi Conjecture (Conjecture 10.5.2) for the compatible circuit decomposition problem.

Problem 1.5.3 (Seymour [210]) Can you prove that a smallest counterexample to the CDC conjecture does not have any cyclic 4-edge-cut?

1.6 Exercises

Exercise 1.1 Explain why the following "proof" to the circuit double cover conjecture is incorrect.

"Proof" Let G be embedded on a surface Σ. Then the collection of the boundaries of all faces (regions) forms a circuit double cover of the graph.

Exercise 1.2 Explain why the following "proof" to the circuit double cover conjecture is incorrect.

"Proof" Let $2G$ be the graph obtained from G by replacing every edge with a pair of parallel edges. Since the resulting graph $2G$ is an even graph, by Lemma A.2.2, the even graph $2G$ has a circuit decomposition $\mathcal{F}$. Is it obvious that $\mathcal{F}$ is a circuit double cover of the original graph G?

Exercise 1.3 Show that the circuit double cover conjecture is equivalent to the following statement.

"Every bridgeless graph has a family $\mathcal{F}$ of circuits such that every edge is contained in one or two members of $\mathcal{F}$." (Open problem (ii) in [119] p. 298.)

Exercise 1.4 Let G be a bridgeless cubic graph. The graph G has a circuit double cover if and only if G has a circular 2-cell embedding on some surface.

Exercise 1.5 Let T be a minimal edge-cut of a graph G with $|T| \leq 3$. Let Q_1, Q_2 be the two components of $G - T$ and for each $\{i, j\} = \{1, 2\}$, let $H_i = G/Q_j$ be the graph obtained from G by contracting all edges of Q_j. If both H_1, H_2 have k-even subgraph double covers (for some integer k), then G also has a k-even subgraph double cover.

Exercise 1.6 (Ding, Hoede and Vestergaard [42]) Let G be a 2-connected graph other than a circuit and with no loop. Assume that G and all of its bridgeless subgraphs have circuit double covers. Show that G has a circuit double cover $\mathcal{F}$ consisting of distinct circuits.

Definition 1.6.1 Let Γ be a group and $S \subset \Gamma$ such that $1 \notin S$ and $\alpha \in S$ if and only if $\alpha^{-1} \in S$. The Cayley graph $G(\Gamma, S)$ is the graph with the vertex set Γ such that two vertices x, y are adjacent in $G(\Gamma, S)$ if and only if $x = y\alpha$ for some $\alpha \in S$.

Exercise 1.7 (Hoffman, Locke and Meyerowitz [109]) Show that every Cayley graph with minimum degree at least 2 has a circuit double cover.

2
Faithful circuit cover

2.1 Faithful circuit cover

The concept of faithful circuit cover is not only a generalization of the circuit double cover problem, but also an inductive approach to the CDC conjecture in a very natural way.

Let $\mathbb{Z}^+$ be the set of all positive integers, and $\mathbb{Z}^\star$ be the set of all non-negative integers.

Definition 2.1.1 Let G be a graph and $w : E(G) \mapsto \mathbb{Z}^+$. A family $\mathcal{F}$ of circuits (or even subgraphs) of G is a faithful circuit cover (or faithful even subgraph cover, respectively) with respect to w if each edge e is contained in precisely $w(e)$ members of $\mathcal{F}$.

Figure 2.1 shows an example of faithful circuit covers of (K_4, w) where $w : E(K_4) \mapsto \{1, 2\}$. Here $w^{-1}(1)$ induces a Hamilton circuit and $w^{-1}(2)$ induces a perfect matching (a pair of diagonals).

It is obvious that *the circuit double cover is a special case of the faithful circuit cover problem that the weight w is constant 2 for every edge.*

Definition 2.1.2 Let G be a graph. A weight $w : E(G) \mapsto \mathbb{Z}^+$ is

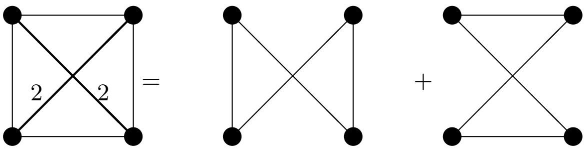

Figure 2.1 *Faithful circuit cover – an example*

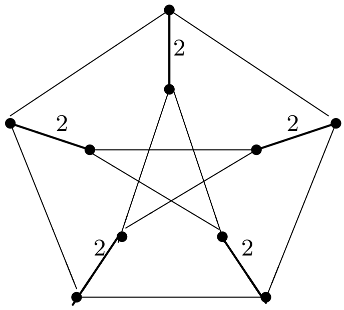

Figure 2.2 (P_{10}, w_{10})

eulerian if the total weight of every edge-cut is even. And (G, w) is called an eulerian weighted graph.

Definition 2.1.3 Let G be a graph. An eulerian weight $w : E(G) \mapsto \mathbb{Z}^+$ is admissible if, for every edge-cut T and every $e \in T$,

$$w(e) \leq \frac{w(T)}{2}.$$

And (G, w) is called an admissible eulerian weighted graph if w is eulerian and admissible.

If G has a faithful circuit cover $\mathcal{F}$ with respect to a weight $w : E(G) \mapsto \mathbb{Z}^+$, then the total weight of every edge-cut must be even since, for every circuit C of $\mathcal{F}$ and every edge-cut T, the circuit C must use an *even* number of *distinct* edges of the cut T. With this observation, the requirements of being *eulerian and admissible* are *necessary* for faithful circuit covers.

Problem 2.1.4 Let G be a bridgeless graph with $w : E(G) \mapsto \mathbb{Z}^+$. If w is admissible and eulerian, does G have a faithful circuit cover with respect to w?

Unfortunately, Problem 2.1.4 is *not* always true. The Petersen graph P_{10} with an eulerian weight w_{10} (see Figure 2.2) does not have a faithful circuit cover: where the set of weight 2 edges induces a perfect matching of P_{10} and the set of weight 1 edges induces two disjoint pentagons (Proposition B.2.26).

Definition 2.1.5 Let G be a bridgeless graph and w be an admissible eulerian weight of G. The eulerian weighted graph (G, w) is a contra pair if G does not have a faithful circuit cover with respect to the weight w.

For a given weight $w : E(G) \mapsto \mathbb{Z}^+$, denote

$$E_{w=i} = \{e \in E(G) : \ w(e) = i\}.$$

Some lemmas and definitions about eulerian weights are presented in this section to prepare for later discussions. Most of these results are straightforward observations.

Proposition 2.1.6 *Let G be a graph and $w : E(G) \mapsto \mathbb{Z}^+$. Then the following statements are equivalent.*

(1) The weight w is eulerian.

(2) The subgraph induced by all odd-weight edges is even.

Proof Exercise 2.3. □

Definition 2.1.7 Let $\mathcal{F}$ be a set of circuits of G. The mapping $w_{\mathcal{F}} : E(G) \mapsto \mathbb{Z}^{\star}$ defined as follows is called the coverage weight of $\mathcal{F}$, or the weight induced by $\mathcal{F}$: for each edge $e \in E(G)$, $w_{\mathcal{F}}(e)$ is the number of members of $\mathcal{F}$ containing the edge e.

Proposition 2.1.8 *Let $\mathcal{F}$ be a set of circuits of G. The weight $w_{\mathcal{F}}$ induced by $\mathcal{F}$ is eulerian and admissible.*

Proposition 2.1.9 *Let G be a graph with a weight $w : E(G) \mapsto \{1, 2\}$. Then w is admissible if and only if G is bridgeless.*

Proof Exercise 2.2. □

Definition 2.1.10 An eulerian weight $w : E(G) \mapsto \mathbb{Z}^+$ is an eulerian $(1, 2)$-weight of G if $1 \leq w(e) \leq 2$ for every $e \in E(G)$.

Since a smallest counterexample to the CDC conjecture is cubic, *most graphs G considered in this book are cubic, and most weights $w : E(G) \mapsto \mathbb{Z}^+$ are eulerian $(1, 2)$-weights.* By Proposition 2.1.9, we may omit the requirement of admissibility if the graph is bridgeless and the $(1, 2)$-weight is eulerian.

The following are some definitions that we will use later.

Definition 2.1.11 Let $w : E(G) \mapsto \mathbb{Z}^{\star}$. The support of w, denoted by $supp(w)$, is the set of edges e with $w(e) \neq 0$.

Definition 2.1.12 Let $w : E(G) \mapsto \mathbb{Z}^{\star}$ be an admissible eulerian weight of G. A family $\mathcal{F}$ of even subgraphs of G is called a faithful k-even subgraph cover if $\mathcal{F}$ is a faithful cover of (G, w) and $|\mathcal{F}| \leq k$.