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Cambridge University Press
978-0-521-28149-2 - Introduction to Dynamics
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1 FIRST-ORDER AUTONOMOUS SYSTEMS

1.1 Basic theory

Dynamics is concerned with the *motion* of systems, that is, their change of state with time t.

A *first-order* system is the simplest type of dynamical system and it is defined by two properties:

(F1) The state of the system is represented by a single real variable x, which may be considered as a coordinate of a point in an abstract one-dimensional space named the *phase space*.

(F2) The motion of the system is represented by a function $x(t)$ of time satisfying a first-order differential equation

$$\frac{dx}{dt} \equiv \dot{x} = v(x, t), \tag{1.1}$$

where v is a given sufficiently well-behaved *velocity function* of x and t, whose value for a particular x and t is the *phase velocity*. The differential equation (1.1) is the *equation of motion* or *equation of change* of the system.

Radioactive decay, population changes in biological species, simple chemical reactions, the fall of a light body through a very viscous fluid and the discharge of an electrical condenser through a resistance are all examples of first-order dynamical systems. Newtonian mechanical systems are not of first order, except in extreme limiting cases like the light body in the viscous fluid, so Newtonian systems will be considered later.

For the rest of this chapter we restrict our attention to first-order *autonomous* systems, because they are particularly simple. An autonomous system is not subject to any external influences that depend on the time, so the velocity function is independent of time and the equation of motion is

$$\dot{x} = v(x). \tag{1.2}$$

The conditions which determine the motion of an autonomous system are independent of time, so we will sometimes refer to it as a system with *time-independent conditions*.

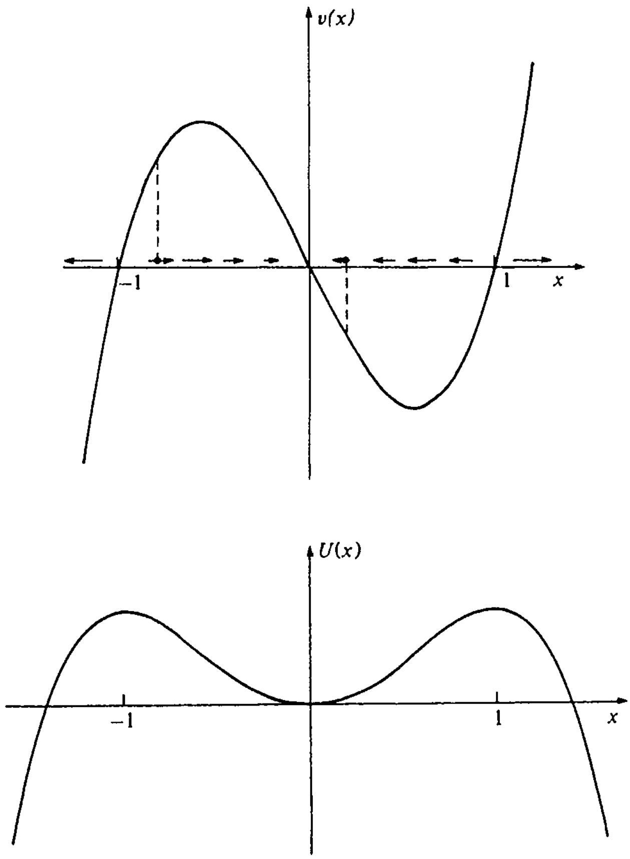
If x_0 represents the state at time t_0, then (1.2) gives t as a function of x:

$$t - t_0 = \int_{x_0}^{x} \frac{\mathrm{d}x}{v(x)}, \tag{1.3}$$

if the integral exists. The inverse function gives x as a function of t. Notice that x depends on t and t_0 only through the time interval $t - t_0$. This property is restricted to autonomous systems.

Without solving equation (1.2) or integrating (1.3), we can obtain the qualitative behaviour of the system graphically, as illustrated in the top half of figure 1.1 in which we represent the velocity function $v(x)$ (we have chosen $v(x) = -x + x^3$ for the sake of example) by a set of arrows as follows:

Fig. 1.1. Graphs of $v(x) = -x + x^3$ and $U(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$ with arrows representing the phase flow.



take a suitable set of values, x_s, of x and, for each x_s, draw an arrow of length proportional to $v(x_s)$ on or near the x-axis, with its *centre* at x_s and pointing in the direction of increasing or decreasing x, depending on the sign of $v(x_s)$.

We can think of a fluid flowing in the phase space with velocity $v(x)$ at all times. The arrows represent the velocity of the fluid, named the *phase flow* and $v(x)$ is its *velocity field*. In this example the fluid is clearly compressible. The changing state of the system is like a particle carried along by the fluid.

The velocity field $v(x)$ of an autonomous first-order system can be expressed as the negative gradient of a potential $U(x)$. Thus

$$v(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{1.4a}$$

where, for some constant U_0,

$$U(x) = U_0 - \int_0^x \mathrm{d}x'\; v(x'). \tag{1.4b}$$

The states of the system flow 'downhill' away from the maxima of the potential $U(x)$, like water flowing down hills and into valleys, as illustrated in the lower half of figure 1.1.

At each zero x_k of the velocity field $v(x)$,

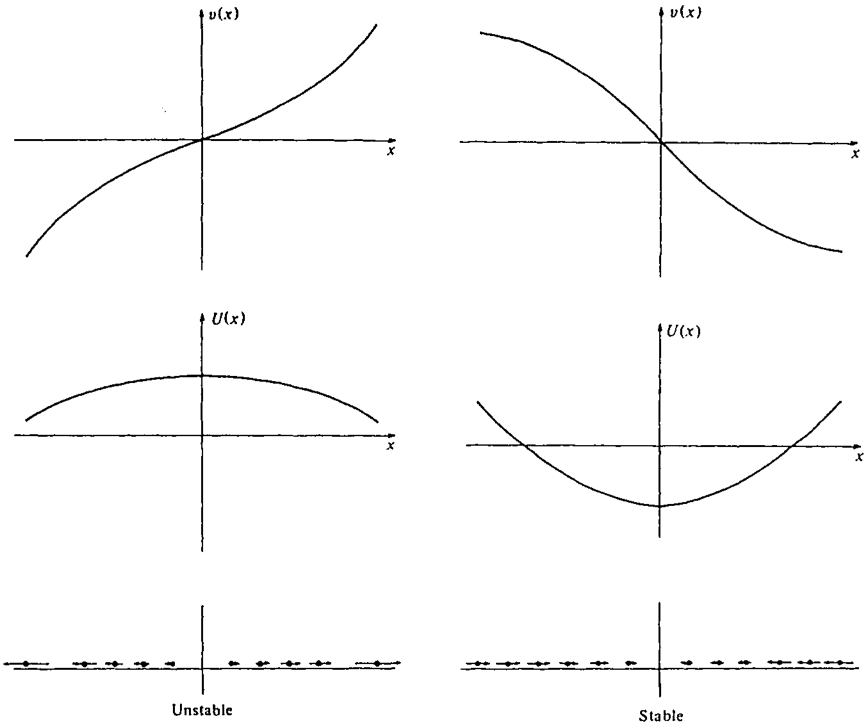
$$v(x_k) = 0, \tag{1.5}$$

so that a system initially at x_k remains there for all time. The points x_k represent states of *equilibrium*: they are named *fixed points*. At all other points the state of the system changes. A system in an open interval between two fixed points cannot pass either of them. Such open intervals, together with those that extend from a fixed point to infinity, are invariant, as are the fixed points. Such fixed points and intervals represent *invariant sets of states* which are defined by the property that if any system is in such a set at some time, then it remains in that set for all times. We usually consider only those elementary invariant sets which cannot be decomposed into smaller invariant sets.

The whole of the phase space of a system is made up of invariant sets, and they provide valuable information about behaviour over arbitrarily long periods of time. The system illustrated in figure 1.1 has three fixed points at $x = 0, \pm 1$, and four elementary one-dimensional invariant sets, which are the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$, bounded on one or two sides by the fixed points.

When the velocity function $v(x)$ has only simple zeros, the fixed points are of two types. There are the *stable* points x_k around which $v(x)$ is a decreasing function of x, so that neighbouring states approach x_k, and there are *unstable* points x_k around which $v(x)$ is an increasing function of x so that neighbouring states leave x_k, as shown in figure 1.2.

Fig. 1.2. Typical velocity fields $v(x)$, potential $U(x)$ and flows in the neighbourhood of stable and unstable fixed points.



Example 1.1
The velocity field is $v(x) = 0$. Every point is a fixed point. The system is always and everywhere at rest,

$$x = x_0. \tag{1.6}$$

Every phase point and every set of phase points is an invariant set.

Example 1.2
The velocity field is $v(x) = a$, $a \neq 0$. If $x = x_0$ at $t = 0$,

$$x = x_0 + at \tag{1.7}$$

with phase diagram

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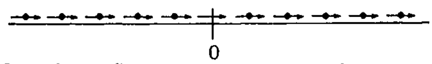
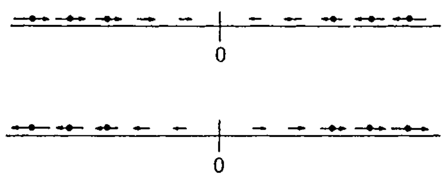
$a > 0$  (1.8)

This is the case of uniform flow, and the entire phase space $(-\infty, \infty)$ is the only invariant set.

Example 1.3
The velocity field is $v(x) = bx$, $b \neq 0$. Then

$$x = x_0 e^{bt} \quad (1.9)$$

with phase diagrams

$b < 0$ / $b > 0$  (1.10)

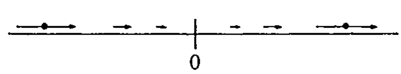
There is one fixed point at $x = 0$, which is stable for $b < 0$ and unstable for $b > 0$.

This example is particularly important as it describes the motion in a sufficiently small neighbourhood of most fixed points. An exception is given by the next example. The first three examples are of *linear* systems, whose motion satisfies a linear differential equation.

Example 1.4
The velocity field is

$$v(x) = cx^2 \quad (1.11a)$$

with phase diagram

$c > 0$  (1.11*b*)

There is one fixed point at $x = 0$, but it is both stable and unstable. Neighbouring states approach it from the left, but leave it from the right. This is because $x = 0$ is a double zero of the velocity function.

The velocity field (1.11) has a property known as *structural instability*. If we take the previous two examples and add to the velocity field a small arbitrary perturbing field $\epsilon\, w(x)$, where $w(x)$ is bounded and differentiable, then, for sufficiently small ϵ, the structure of the invariant sets remains unchanged. But if we do the same for the field (1.11), for example with $w(x) = 1$, the number of zeros of $v(x)$ is increased to 2 or decreased to 0, however small ϵ may be. A system which retains its structure under such a small perturbation is *structurally stable*. Otherwise it is structurally unstable, like example 1.4.

Care must be taken to distinguish the stability of states, or phase points, from the structural stability of systems, or differential equations. For simplicity we usually consider structurally stable systems, because they are typical.

The system with velocity field (1.11) also illustrates another important feature, *terminating motion*. Suppose at time $t = 0, x = x_0 > 0$. Then the solution of the differential equation of motion gives

$$x = (x_0^{-1} - ct)^{-1}, \quad t < (cx_0)^{-1}. \tag{1.12}$$

The motion terminates abruptly at the critical time $t = (cx_0)^{-1}$ when x tends to infinity, and is undefined beyond that time. Not all velocity fields define the motion of the system for all time. In practice the validity of the equation of motion itself breaks down before the critical time is reached. For $x_0 < 0$ there is no positive critical time, but an attempt to determine the *past* motion of the system leads to a similar difficulty.

In general, we refer to the motion as *terminating* if it is represented by a solution of the equation of motion which is undefined at any finite point of the real time axis. Sometimes, but not always, this happens when x becomes infinite. Terminating motion is typical of systems in the exterior invariant intervals of certain polynomial velocity fields. It occurs whenever $v(x)$ is a polynomial of degree two or greater.

1.2 Rotation

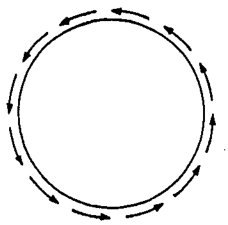
The phase space does not always occupy the whole real line. For some types of motion, typified by rotation about an axis, the phase space can be considered as a circle. In that case it is convenient to choose a coordinate θ in the range $[0, 2\pi]$ and to identify the ends of the range. The equation of motion for an autonomous system is

$$\dot{\theta} = v(\theta), \tag{1.13}$$

where $v(\theta)$ must be a periodic function of θ with period 2π, $v(\theta) = v(\theta + 2\pi)$. For these systems it is possible to have bounded motion with no fixed points, for example

$$\dot{\theta} = \omega = \text{constant}, \tag{1.14a}$$

which represents uniform rotation with period $T = 2\pi/\omega$.



(1.14*b*)

Any circular motion with no fixed point is named a *rotation* and has a period T given by

$$T = \int_{\theta=0}^{\theta=2\pi} \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}\theta}{v(\theta)}. \qquad (1.15)$$

1.3 Natural boundaries

If the variable x represents the distance of a particle from a point in space, or the population of a large number of objects such as living cells or radioactive atoms, then negative values of x have no meaning. The phase space has a *natural boundary* at $x = 0$. Usually it helps to include the boundary point in the phase space. Normally, for first-order systems, if the motion does not terminate, the velocity at a natural boundary is zero.

For first-order autonomous systems examples of velocity fields with natural boundaries are given by taking any velocity field with fixed points and restricting the phase space to an invariant set bounded by, and including, a fixed point or points. For example, (1.10), excluding the negative real axis, represents exponential decay of a population, with a natural boundary at $x = 0$.

Example 1.5 A system that terminates at its natural boundary.
The equation of motion is

$$\dot{x} = v(x) = -\sqrt{x} \qquad (x \geqslant 0), \qquad (1.16)$$

where the square root is taken as positive. The system has a natural boundary at $x = 0$, and x always decreases. The general solution of the equation of motion is given by $\mathrm{d}x/\sqrt{x} = -\mathrm{d}t$; it is

$$2\sqrt{x} = C - t,$$

where C is a constant, so

$$x = \tfrac{1}{4}(C-t)^2, \qquad (t \leqslant C). \qquad (1.17)$$

The system reaches the natural boundary at $t = C$. Later than this, equation (1.17) does not satisfy (1.16) but is a solution of

$$\dot{x} = \sqrt{x}, \tag{1.18}$$

with the opposite sign. No motion satisfying (1.16) is possible beyond $t = C$. However equation (1.17) defines a possible motion in which the system satisfies the equation (1.16) for $t \leqslant C$ and (1.18) for $t \geqslant C$. Hamiltonian systems with conserved quantities sometimes move in this way.

More general systems and theorems of this type appear in exercises 1.8 and 1.9.

1.4 Examples from biology

The dynamics of biological populations is a branch of ecology. The populations of insects, birds, fish and mammals are increased by birth and decreased by death. These processes depend on many factors that are less well understood than the factors that influence the motion of levers, pulleys, projectiles and planets. Nevertheless, simplified models can help us to understand the way that these populations change with time.

When the population of a single species in some specified region is sufficiently large it may be represented by a continuous variable x. If the birth and death rates per individual of the populations, $B(x)$ and $D(x)$, depend on the population x, but not on space, time, or any other factors, then the population is a first-order autonomous system with equation of motion $\dot{x} = v(x)$ and velocity function

$$v(x) = [B(x) - D(x)]\, x \quad (B(x) \geqslant 0,\ D(x) \geqslant 0). \tag{1.19}$$

For populations, $x \geqslant 0$, so that $x = 0$ is always a natural boundary.

Example 1.6 Exponential growth and decline

The simplest assumption is that the birth rate B and death rate D are constants independent of the population. The equations are those of example 1.3 with $b = B - D$. If we exclude the special case when $B = D$, the natural boundary at $x = 0$ is the only fixed point. If D exceeds B then the population decays exponentially to zero. If B exceeds D, the population increases exponentially without bound. The phase diagrams are given by the right halves of (1.10).

Example 1.7 The logistic equation

In practice the population in a confined region cannot increase without bound forever, as there are limiting factors, such as competition for food and living space. The simplest assumption is to suppose that these factors leave the birth rate unchanged, but produce a death rate per individual proportional to the population, so we can write

$$B(x) = b, \quad D(x) = cx \quad (x \geqslant 0, b > 0, c > 0) \tag{1.20}$$

and the equation of change has the form

$$\dot{x} = bx - cx^2. \tag{1.21}$$

This is known as the *logistic equation* and many actual populations closely follow it. It is not difficult to solve, or to analyse, so this is given as exercise 1.13. It gives a stable population.

However, the situation is rarely so simple. Frequently one species preys on another, so that their population equations are coupled together, leading to systems of second order as in chapter 3.

Sometimes an individual species has a definite reproductive season, so that the change in population is not represented by a differential equation, but by a difference equation or map. This is considered briefly in chapter 11. In both cases completely new and remarkable phenomena appear in the time dependence of the populations.

Exercises for chapter 1

(1) Three first-order systems have the following velocity functions. Which of them is autonomous?

(*a*) $v(x, t) = e^x$;
(*b*) $v(x, t) = t$;
(*c*) $v(x, t) = \begin{cases} 0 & (t < 0) \\ x^2 & (t \geqslant 0) . \end{cases}$

(2) Draw the phase diagrams and find the fixed points and invariant sets of systems with the following velocity functions:

(*a*) $v(x) = (a - x)(x - b) \quad (-\infty < x < \infty, b > a > 0)$;
(*b*) $v(x) = (a - x)(b - x) \quad (-\infty < x < \infty, b > a > 0)$.

Without solving the equations of motion, discuss the qualitative behaviour after $t = 0$ if $a < x(0) < b$ in each case.

(3) The angle of a blade of a food mixer is denoted by ψ and its motion is determined by the differential equation

$\dot{\psi} = a + b \sin \psi \quad (0 \leqslant \psi \leqslant 2\pi, a > 0, b > 0).$

Find the fixed points and invariant sets of the motion when $a < b$, $a = b$ and $a > b$. For what values of a and b is the motion a rotation? Find the period.

(4) Is the system of example 1.1 structurally stable? Give reasons for your answer.

(5) For what positive values of a and b is the system with velocity function

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$$v(x) = (a - x)(x - b) \quad (-\infty < x < \infty, b > 0, a > 0)$$

structurally stable?

(6) For what values of a and b is the food mixer blade of exercise 1.3 structurally stable?

(7) Discuss the nature of the motion of the systems with velocity functions

(*a*) $v(x) = x \sin x \quad (-\infty < x < \infty)$,
(*b*) $v(x) = x \cos x \quad (-\infty < x < \infty)$,

and determine which of them, if any, is structurally stable.

(8) For what positive values of α does the motion with velocity function

$$v(x) = x^\alpha \quad (x > 0, \alpha > 0)$$

terminate?

Exercises for Hamiltonian mechanics (all square roots are non-negative)

(9) State the important features of the motion of the systems with velocity functions

(*a*) $v(x) = a\sqrt{x - b} \quad (a > 0, x \geqslant b)$,
(*b*) $v(x) = \sqrt{1 - x^2} \quad (|x| \leqslant 1)$.

(10) (*a*) Suppose that $A < B$ and that

$$0 < v_1(x) \leqslant v_2(x) \quad (A \leqslant x \leqslant B)$$

$v_1(B), v_2(B)$ finite.

Show that, if T_i is the time that system S_i with velocity function v_i takes to go from state $x = A$ to state $x = B$, then

$$T_2 \leqslant T_1.$$

(*b*) Hence show that, if $f(x)$ and $f'(x)$ exist in an interval around $x = B$ with values satisfying

$$f(B) = 0, f'(B) < 0,$$

then the motion defined by

$$v(x) = \sqrt{f(x)}$$

terminates at $x = B$.

(*c*) Describe the motion with velocity function

$$w(x) = \sqrt{g(x)},$$

where $g(x)$ is a polynomial with N distinct real zeros.

Exercises on chemical reactions

(11) Nitric oxide (NO) and oxygen (O_2) react to form NO_2 as follows

$$2NO + O_2 \to 2NO_2.$$