Part 1: 3-manifolds

The classification of compact 3-manifolds

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The aim of this article is to summarise the basic facts known about the classification of compact 3-manifolds, and to explain briefly how Thurston's recent work fits in and adds to these facts. Essentially, this note forms the background needed to appreciate the conjecture at the beginning of my notes of Thurston's lectures (which follow this article). Hempel's book [1] is the reference for all results stated here for which a specific reference is not given.

Our starting point is a theorem of Kneser proved in the 1930's, about connected sums of compact 3-manifolds. Call a 3-manifold M *prime* if for any decomposition $M = M_1 \# M_2$ as a connected sum, one of M_1, M_2 is homeomorphic to S^3. Kneser proved that a compact 3-manifold can always be expressed as a connected sum of prime manifolds. Milnor added to this the information that if M is also orientable, then the prime summands of M are unique. If M is non-orientable, this need not be the case but the only examples here are similar to those in surface theory. For example if F is a non-orientable surface and T and K denote the torus and Klein bottle, then $F \# T$ is homeomorphic to $F \# K$. Similarly, if F is a non-orientable 3-manifold, and $S^1 \tilde{\times} S^2$ and $S^1 \times S^2$ denote the non-trivial and trivial S^2-bundles over S^1, then $N \# (S^1 \tilde{\times} S^2)$ is homeomorphic to $N \# (S^1 \times S^2)$.

The preceding paragraph shows that one can reduce the classification problem for compact 3-manifolds to that for prime manifolds. At this point, one needs to make another definition. We will say that a 3-manifold M is *irreducible* if any embedded 2-sphere in M bounds a 3-ball. Clearly an irreducible manifold is prime. The converse is not true but the only 3-manifolds which are prime and not irreducible are the two S^2-bundles over S^1 mentioned above. Thus we can restrict our attention to irreducible manifolds.

If M is orientable and irreducible, then the Sphere Theorem at once implies that $\pi_2(M) = 0$. It now follows easily that the universal covering of M is contractible or is a homotopy 3-sphere Σ. From this, it follows that M is aspherical with infinite, torsion free fundamental group, or M is D^3, or M is finitely covered by a homotopy 3-sphere. This last possibility gives rise to many extremely hard problems of which the Poincaré Conjecture is just one.

If M is non-orientable and is P^2-irreducible, i.e. M is irreducible and contains no two-sided projective planes, then the Projective Plane Theorem implies that $\pi_2(M) = 0$, so that again the universal covering of M is contractible or is a homotopy 3-sphere Σ. The second case cannot occur as any orientation reversing homeomorphism of Σ must have a fixed point and so cannot be a covering transformation.

Finally if M is non-orientable and irreducible but does admit two-sided projective planes, then there is a canonical family of disjoint, non-parallel, two-sided projective planes in M which cut M into pieces in each of which any two-sided projective plane must be parallel to a boundary component. Thus all the pieces we obtain have at least one boundary projective plane. If we take the orientable double covering of such a piece and glue a 3-ball to each boundary sphere, the resulting manifold M_1 will be irreducible.

We conclude from the above that any compact 3-manifold can be cut up by 2-spheres and projective planes in a way which is very nearly canonical into pieces which after gluing 3-balls to all boundary spheres, cannot be further cut up. Note that many of these pieces are aspherical. This fact explains the very close connection between 3-dimensional topology and combinatorial group theory.

We now consider the compact orientable irreducible case. If $\pi_1(M)$ is finite, M is the 3-ball or is covered by a homotopy 3-sphere. Apart from the 3-ball, all known examples are Seifert fibre spaces. If the Poincaré Conjecture holds and if all free finite group action on S^3 are equivalent to standard actions, then these examples are the only possible examples. On the other hand when $\pi_1(M)$ is infinite, Seifert fibre spaces are rather unusual. For example, the only knots whose complements are Seifert fibred are torus knots. A very nice class of manifolds to consider is the class of Haken manifolds. Recall that a surface F, not S^2, properly embedded in a 3-manifold M in a two-sided manner, is *incompressible* if the induced map

$\pi_1(F) \to \pi_1(M)$ is injective. A manifold M is *Haken* if there is a sequence of 3-manifolds $M = M_0, M_1, \ldots, M_n$ and surfaces $F_0, F_1, \ldots, F_{n-1}$ such that F_i is an incompressible surface in M_i, M_{i+1} is obtained from M_i by cutting along F_i and M_n is a disjoint union of 3-balls. This sequence is called a *hierarchy*. Waldhausen proved that any compact, orientable, irreducible 3-manifold with non-empty boundary is Haken. However there are many examples of closed, orientable, irreducible 3-manifolds with infinite fundamental group which are not Haken. The known examples are all Seifert fibre spaces or have a complete hyperbolic structure so they do not provide counter examples to Thurston's conjecture. One of the specially nice features of Haken manifolds is that they tend to be characterised by their fundamental group. For closed Haken manifolds this is exactly true, i.e. closed Haken manifolds with isomorphic fundamental groups are homeomorphic. For manifolds with boundary, there are analogous results but they are more complicated to state. In a sense, this reduces the classification problem for Haken manifolds to group theory, but in general it is hard to tell if two groups are isomorphic and there is also no useful group theoretic characterisation of the fundamental groups of Haken manifolds.

As I said before, there are examples of closed orientable irreducible 3-manifolds with infinite fundamental group which are not Haken. However, the Seifert fibre space examples are known to have a finite covering space which is Haken. This led Waldhausen to the following

CONJECTURE *Any closed orientable irreducible 3-manifold with infinite fundamental group has a finite covering space which is Haken.*

This conjecture holds for some of the known hyperbolic non-Haken examples, but has not been proved for all the known examples. It is not even known if all of these examples have closed surface groups contained in their fundamental group. In the general case, when a hyperbolic structure is not assumed, it is not even known that the universal covering space $\tilde{M}$ of a non-Haken manifold M must be homeomorphic to $\mathbb{R}^3$. The main problem here is that the ends of the universal covering space are not known to be tame. Until recently, there was also the problem that $\tilde{M}$ might contain fake balls. But this has now been shown to be impossible by Yau using minimal surface theory.

The class of Haken manifolds is extremely large but the sub-class of Haken Seifert fibre spaces is small and completely under-stood. The work of Johannson [3] and of Jaco and Shalen [2]

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shows that any Haken manifold has a Seifert fibred part and a non-Seifert fibred part. In fact they proved much more, but for the purposes of this note we shall concentrate on this particular result. If a compact Haken manifold M has compressible boundary, then the Loop Theorem gives a 2-disk D properly embedded in M. Let N be the manifold obtained by cutting M along D. Thus M is obtained from N by attaching a 1-handle to ∂N. This simple description allows us to consider only Haken manifolds with incompressible boundary. All other Haken manifolds are obtained by adding 1-handles. Johannson [3] and Jaco and Shalen [2] show that if M is a Haken manifold with incompressible boundary, then there is a canonical family of disjoint incompressible tori which cuts M into pieces which are either Seifert fibred spaces or are atoroidal. The atoroidal Haken manifolds are exactly the manifolds whose interiors Thurston tells us admit a complete hyperbolic structure.

The preceding discussion of Haken manifolds was for the orientable case only. In the non-orientable case the situation is easier in some respects. In particular any non-orientable, compact, P^2-irreducible 3-manifold is Haken. As in the orientable case, one can also restrict attention to manifolds with incompressible boundary. The orientable double covering $\tilde{M}$ of such a manifold M is automatically Haken and will again have incompressible boundary. Hence the Torus Splitting Theorem mentioned above holds for $\tilde{M}$. Now it is not hard to deduce that the canonical family of tori in $\tilde{M}$ can be isotoped to the invariant under the covering involution. Hence, M has a canonical family of tori and Klein bottles which cuts M into pieces which are either atoroidal or are double covered by Seifert fibre spaces. The conclusion of the preceding paragraphs taken together with Thurston's hyperbolisation theorem is that any Haken manifold can be cut canonically into Seifert fibre space pieces and hyperbolic pieces. Now the fundamental group of a hyperbolic piece is a sub-group of $PSL(2, \mathbb{C})$ and this gives much new information about 3-manifold groups. For example, it follows that the fundamental group of a Haken manifold is residually finite. But even more importantly, the hyperbolisation result tells one to use geometric ideas as opposed to purely topological ideas when considering 3-manifolds. For example, the question of when a covering space of a Haken manifold can be compactified has now been answered by Thurston in many cases. Thus, although it is not clear that a classification of 3-manifolds is closer, Thurston's work has already had a great impact on 3-dimensional topology and on Kleinian group theory, and is going to lead to a much deeper understanding of both subjects.

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Hyperbolic geometry and 3-manifolds

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My theme is that geometrical methods yield more information about manifolds than do purely topological methods and in dimension three geometric methods are often applicable. A good example of this is the recent proof of the Smith Conjecture.

Here is the strongest possible conjecture asserting that one can always use geometry when studying 3-manifolds. The Poincaré Conjecture is a very special case.

CONJECTURE. *Every compact 3-manifold* M *with incompressible boundary has a canonical decomposition into geometric pieces, i.e. by cutting* M *along a canonical family of disjoint, 2-sided, closed surfaces each homeomorphic to* S^2, P^2, T^2 *or the Klein bottle, one can obtain geometric pieces.*

When I say that a 3-manifold M is *geometric*, I mean that the interior of M has a complete geometric structure modelled on some homogeneous space. By a *homogeneous space*, I shall mean a space X and a transitive group G of homeomorphisms of X with the property that G_x, the stabiliser of x, is compact for every x in X. It follows that X admits a G-invariant metric. We will always assume that X is equipped with such a metric and we will usually assume that G is maximal i.e. the full isometry group of X.

A manifold M without boundary has a (X,G)-structure if it is locally homeomorphic to open subsets of X and there is an atlas of charts such that all the overlap maps lie in G. Such a manifold inherits a metric from the metric on X.

If X is simply connected and M has a complete (X,G)-structure, then M is isometric with the quotient of X by some subgroup Γ of G acting as a group of covering transformations of X. In particular Γ is isomorphic to $\pi_1(M)$. Of course, a homogeneous space (X,G), and hence any manifold with (X,G)-

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structure, has constant Gauss curvature.

The following result is a partial affirmation of the conjecture.

THEOREM 1. *The conjecture holds for Haken manifolds.*

The more usual formulation of the result is that one can cut an orientable Haken manifold with incompressible boundary along tori to obtain pieces which are Seifert fibre spaces or admit a complete hyperbolic structure. This gives a somewhat misleading picture as there are many relevant geometries apart from hyperbolic geometry. It just happens that most of those geometries give rise to Seifert fibre spaces.

Note that if M is a Haken manifold with compressible boundary one can cut M along 2-discs to obtain a Haken manifold with incompressible boundary. Thus M also admits a decomposition into geometric pieces. However, in general, this decomposition is not canonical as the family of 2-discs is not canonical, e.g. if M is a handlebody.

The first step in approaching the conjecture is to understand which types of geometry can occur. Thus one wants to classify simply connected Riemannian manifolds with transitive isometry groups, and hence constant Gauss curvature.

In dimension 2, it is easy to classify simply connected homogeneous spaces by their Gauss curvature K. By an appropriate scale change one can assume that K is $-1, 0$ or 1. The three corresponding homogeneous spaces are H^2, E^2 and S^2. A nice fact here is that the isometry group is transitive on directions as well as points, so that the three spaces have constant curvature, i.e. constant sectional curvature. This does not hold in dimension three. In the case $K = 0$, the torus is the only closed orientable surface admitting a $(E^2, \mathrm{Isom}(E^2))$ structure. These structures give the flat tori. We now fix a torus T and an element α of $\pi_1(T)$ which can be represented by an embedded loop in T. Given a flat structure on T, we first make a scale change so that the area of T is 1. There is a closed geodesic λ (which is not unique) which represents α. Cut T along λ and let ℓ be the length of λ. One obtains a cylinder C of height ℓ^{-1} and circumference ℓ. This cylinder is completely determined up to isometry by the number ℓ. Hence the flat structure on T is determined by ℓ and by a real number θ which measures the twist used to glue the two boundary circles of C. Hence the space of flat structures on T is homeomorphic to $\mathbb{R}^2$, with a particular structure having coordinates $(\log \ell, \theta)$.

In the case $K < 0$, there is a similar description for the space of closed orientable surfaces of a given genus with hyperbolic structure. This is the Teichmüller space of the surface. One cuts the surface of genus g, M say, along $3g - 3$ closed geodesics in the homotopy classes shown in Fig. 1 so that M falls into $2g - 2$ pairs of pants. Each pair of pants is determined by the lengths of its three boundary curves, and M is determined by these lengths together with a twist parameter for each of the circles. Thus the Teichmüller space of M has dimension $6g - 6$.

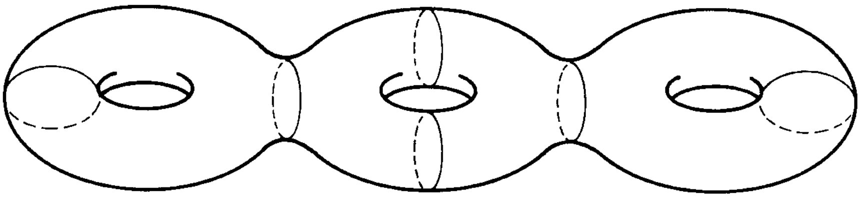

Fig. 1

Now we turn to the 3-dimensional homogeneous spaces. There are eight relevant simply connected ones. *Relevant* means that there is a manifold with a complete (X,G)-structure which has finite volume. An example of an irrelevant homogeneous space is obtained by taking X to be $\mathbb{R}^3$ and G to be generated by homeomorphisms of the form

$$(x,y,z) \to (x+r,\ y+s,\ z)\ ,\quad \forall r,s \in \mathbb{R}$$

$$(x,y,z) \to (e^{-t}x,\ e^{-t}y,\ z+t)\ ,\ \forall t \in \mathbb{R}\ .$$

We obtain the classification of the 3-dimensional homogeneous spaces by considering the size of G_x. For convenience we will take G to be a group of orientation preserving isometries.

Case $G_x = SO_3$

In this case X has constant curvature and, as in the 2-dimensional case, we obtain the examples H^3, E^3 and S^3 with curvature $-1, 0, 1$. In each case, if G denotes the full orientation preserving isometry group of X, there is a bundle $SO_3 \to G \to X$.

Case $G_x = SO_2$

Let T_x be the tangent space at x. Then $T_x = L_x \oplus P_x$ where L_x is the line fixed by G_x and P_x is the orthogonal plane which is invariant under G_x. As X is simply connected, we can choose coherent orientations on the L_x, so as to obtain a unit vector field on X. This vector field and the plane field P_x are both G-invariant and so descend to any manifold covered by X. The flow on X determined by the vector field also leaves the plane field invariant. This flow must preserve the area of the planes. This is because X has a quotient M of finite volume, by hypothesis, and the flow on M must preserve volume. It then follows that the flow on X must preserve the metric on the planes of the plane field. For if some direction in a plane were contracted or expanded by the flow, then so would every direction expand or contract, and the flow could not preserve area.

There are six cases according to whether the planes of the plane field have Gauss curvature $k > 0$, $k = 0$ or $k < 0$ and according to whether the plane field is integrable or not. Two of the cases, S^3 and E^3, have appeared already, so we have a total of seven homogeneous spaces so far. The six cases are

	$k > 0$	$k = 0$	$k < 0$
Integrable	$S^2 \times E$	$E^2 \times E^1 = E^3$	$H^2 \times E$
Not integrable	S^3	N	$\widetilde{SL_2(\mathbb{R})}$

where, N is the nilpotent Lie group of dimension three and $\widetilde{SL_2(\mathbb{R})}$ is the universal cover of $SL_2(\mathbb{R})$. There is an exact sequence $0 \to \mathbb{R}^1 \to N \to \mathbb{R}^2 \to 0$, so the four spaces in the case $k \leq 0$ are all topologically $\mathbb{R}^3$. The space N gives the correct geometry for non-trivial circle bundles over the torus.

Case $G_x = 1$

In this case, $X = G$ and we have a Lie group. The only new possibility is the solvable Lie group S given as a split extension $0 \to \mathbb{R}^2 \to S \to R \to 0$, where t in $\mathbb{R}$ acts on $\mathbb{R}^2$ by the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

The compact 3-manifolds obtained from this homogeneous space